2018-2019 – Trento Summer School

Cycles in Business Cycles

Franck Portier
University College London
f.portier@ucl.ac.uk

Version 1.0
2/06/2019

Notes

- Based on my work with Paul Beaudry & Dana Galizia
- I will not take time for references, other work, etc
- Check on my webpage for papers and references
- Write me : f.portier@ucl.ac.uk

Objective of the lecture

- Show that there is *cyclicality* (to be defined) in economic fluctuations. (data)
- Show that the economy might be thought as fluctuating around an *unstable* steady state. (data)
- Discuss how such fluctuations can be seen as an emergent phenomenon in a environment with interactions (theory)
- Show a fully micro-founded-general-equilibrium-rational-expectations model that can be solved and estimated (theory and data)
- Under study : developed economies.

Roadmap

Roadmap

I. Cyclicality
II. Instability
III. Generating Cycles through Dynamic Models with Interactions (theory)
IV. A Fully Specified Model

I. Cyclicality

- Cycles are "recurrent movements in economic activity"
- Booms and busts
- Can be thought as the consequence of shocks hitting an otherwise stable economy...
- ... Or as the very indication that that market (capitalist) economies are intrinsically unstable.
- Let's try to see what's in the data.
- Start with the NBER series of 1 and 0 for expansions and recessions.

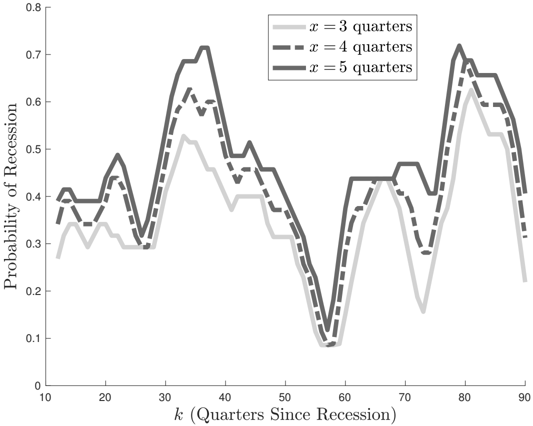
I. Cyclicality

TABLE 1 – Recent U.S. Business Cycles, as identified by the NBER's Business Cycle Dating Committee

BUSINESS CYCLE REFERENCE DATES		**DURATION IN MONTHS**			
Peak	**Trough**	**Contraction**	**Expansion**	**Cycle**	
Quarterly dates are in parentheses		*Peak to Trough*	*Previous trough to this peak*	*Trough from Previous Trough*	*Peak from Previous Peak*
April 1960(II)	February 1961 (I)	10	24	34	32
December 1969(IV)	November 1970 (IV)	11	106	117	116
November 1973(IV)	March 1975 (I)	16	36	52	47
January 1980(I)	July 1980 (III)	6	58	64	74
July 1981(III)	November 1982 (IV)	16	12	28	18
July 1990(III)	March 1991(I)	8	92	100	108
March 2001(I)	November 2001 (IV)	8	120	128	128
December 2007 (IV)	June 2009 (II)	18	73	91	81

I. Cyclicality

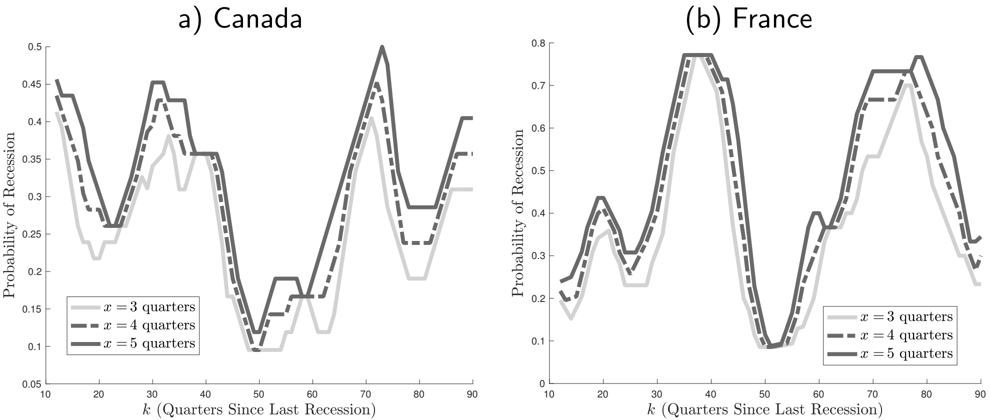
Conditional Probability of Being in a Recession (US)



Notes : This shows the fraction of time the economy was in a recession within an x-quarter window around time $t + k$, conditional on being in a recession at time t, where x is allowed to vary between 3 and 5 quarters.

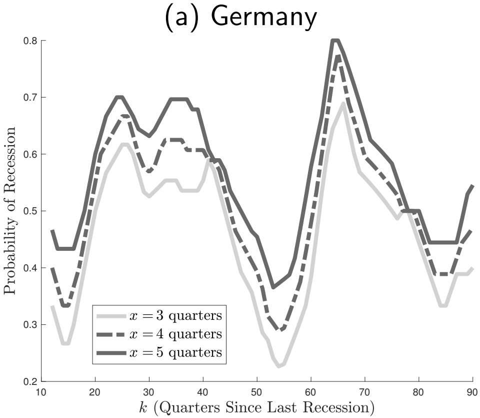
I. Cyclicality

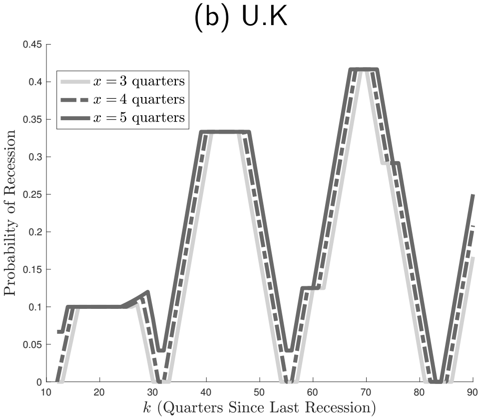
Conditional Probability of Being in a Recession



I. Cyclicality

Conditional Probability of Being in a Recession

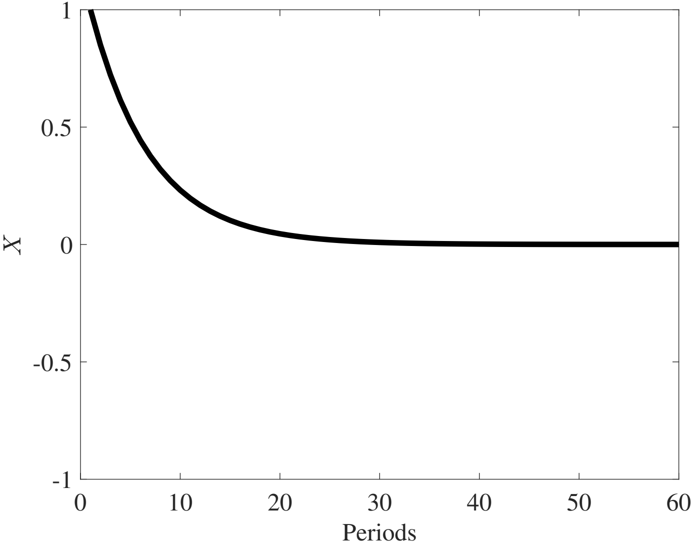


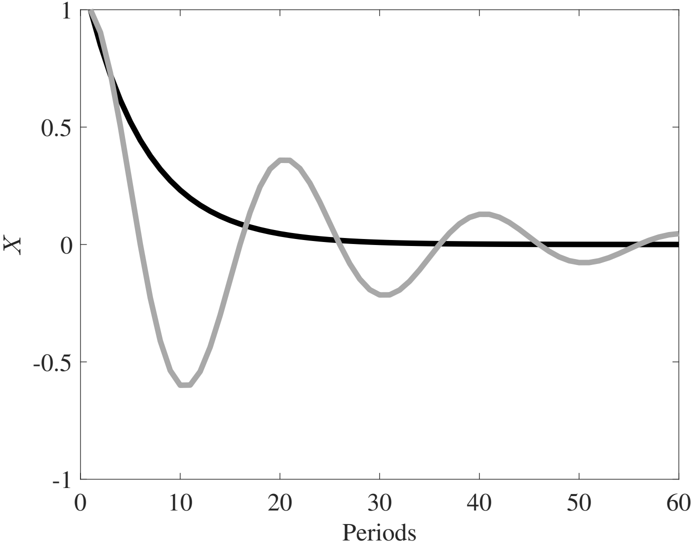


- What is meant by cyclicality ?
  - If activity is high today,
  - at say $N/2$ period in the future, economic activity is expected to be low (below trend),
  - and then at N expected to be high again and so on.
- This translates in cyclicality in the auto-covariance
- Note : *nothing deterministic* about this definition, its only about conditional expectations.
- Different from the more standard "auto-regressive" (AR(1)) view.
  - If activity is high today,
  - we expect it to return to mean.
- The two views differ on whether or not we should worry about big booms.

I. Cyclicality

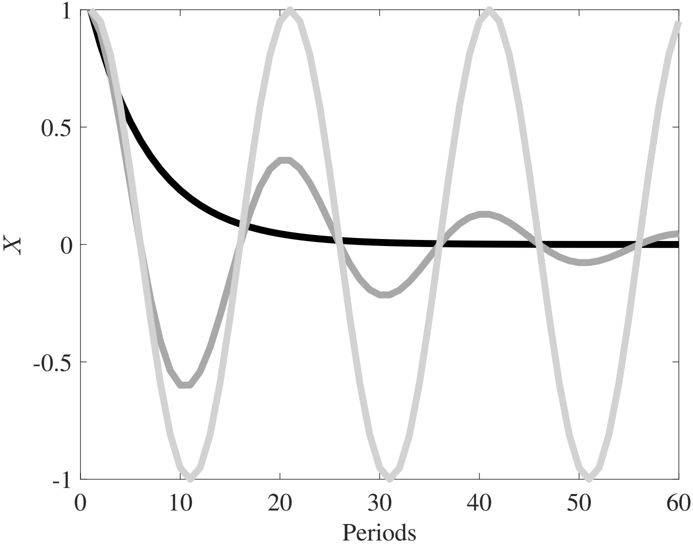
Absence of Cyclicality




1
0.5
0
-0.5
-1
X
0
10
20
30
40
50
60
Periods

I. Cyclicality

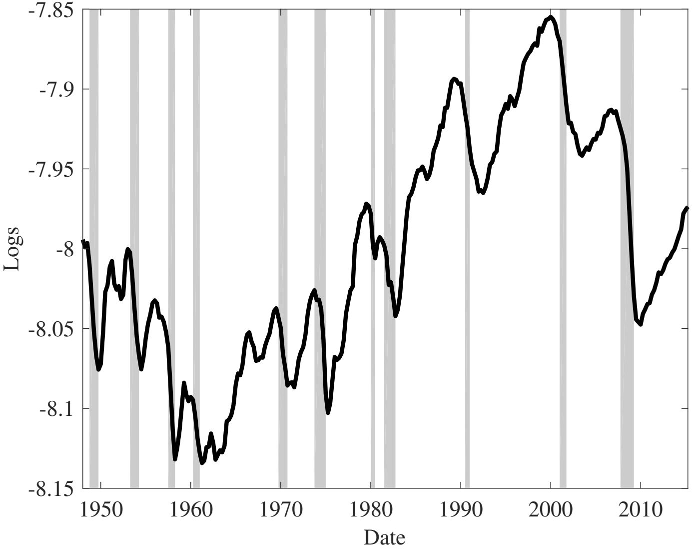
"Strong" Cyclicality

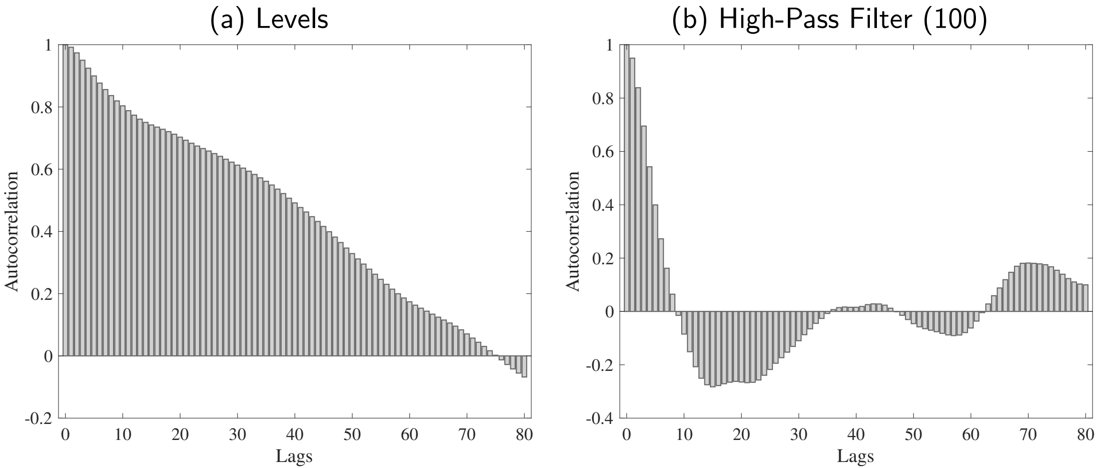


- Issue seems easy to settle : just look at auto-covariance function of the data
- Difficulty : Many macro variables are trending, so this requires a trend-cycle decomposition
- But such decompositions can create spurious cycles
- Solution : Look at hours worked per capita (non trending)

I. Cyclicality

Non-Farm Business (NFB) Hours Per Capita




(a) Levels
Autocorrelation
Lags
1
0.8
0.6
0.4
0.2
0
-0.2
0
10
20
30
40
50
60
70
80
(b) High-Pass Filter (100)
Autocorrelation
Lags
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
10
20
30
40
50
60
70
80

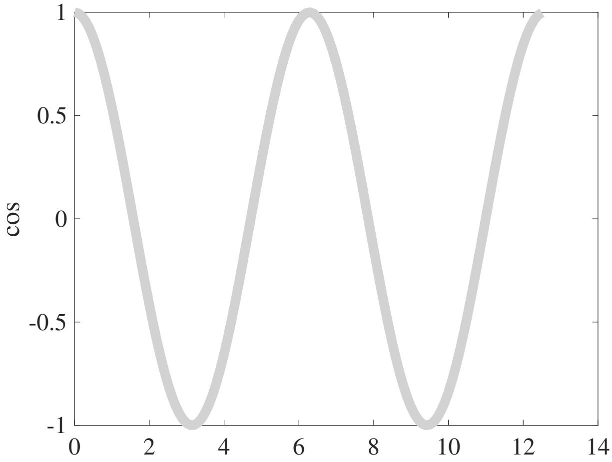
- A different (better) way to look at cyclicality is to look at spectral density
- There is a one to one mapping between autocovariance and spectral density
- but spectral density "weights" autocorrelations with their contribution to total variance
- $\rightsquigarrow$ autotocorrelations at longer horizons are "boosted" and separated from shorter horizons.

- Idea : A series can be seen as the sum of periodic functions.
- A typical periodic function is $\cos(\omega t)$, with period (the time it takes to reproduce itself) $2\pi/\omega$.
  - × Knowing that period of $\cos(t)$ is 2π, for a given t_1, what is the t_2 such that $\cos(\omega t_2) = \cos(\omega t_1)$?
  - × The solution is $t_2 - t_1 = 2\pi/\omega$.
- $\frac{\omega}{2\pi}$ is the *frequency* of oscillation (number of cycles per unit of time)

I. Cyclicality

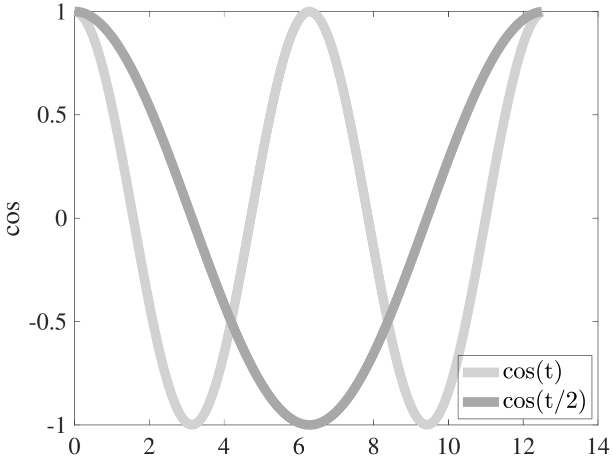
Typical periodic functions

FIGURE 1 – Cosine wave with ω=1



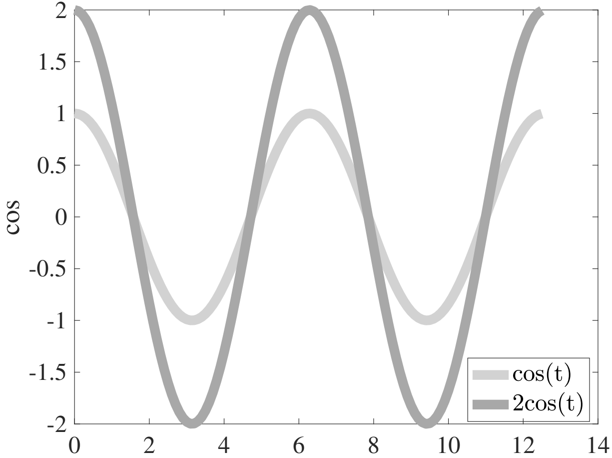
- With $\omega = 1$, the period is $2\pi = 6.28$ and frequency is $\frac{1}{2\pi} = 0.16$.

FIGURE 2 – Cosine waves with ω=1 or 1/2



- With $\omega = 1/2$, the period is $4\pi = 12.56$ and frequency is $\frac{1}{4\pi} = 0.08$.

FIGURE 3 – Cosine waves with ω=1 and different amplitudes

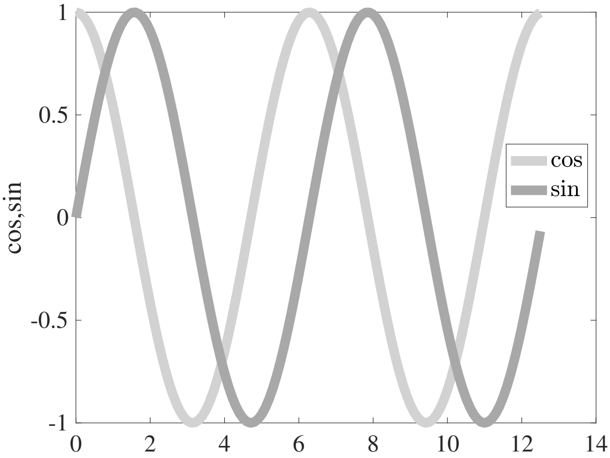


- Here are plotted $A\cos(t)$ with $A = 1$ or $A = 2$.

I. Cyclicality

Decomposing a time series into frequency domain

FIGURE 4 – Cosine and Sine waves with ω=1



- $\sin(\omega t)$ behaves the same way, with same amplitude and period, but with a phase shift

- The idea of spectral decomposition is that with sin and cos, we can span the whole space of covariance stationary time series : the typical periodic function is

$$a\cos(\omega t) + b\sin(\omega t) \tag{1}$$

whose period is $2\pi/\omega$ but whose phase and amplitude depend on (a, b)

- Here we want to treat a and b as mean zero random variables.
- There is always a sum of type (1) periodic functions that reproduces a given time series
- The spectral density or *spectrum* of a series indicates the weight of each frequency (from low to high) in the total variance of the series.

I. Cyclicality

Decomposing a time series into frequency domain

- A (second order) stationary time series x_t with $E[x_t] = 0$ has three fundamental representations :
- 1. Autocovariance function

$$\lambda_\tau = E[x_t x_{t-\tau}]$$

- 2. MA representation (WOLD theorem) :

$$x_t = \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \qquad \text{and} \qquad \lambda_\tau = \left(\sum_{j=0}^{\infty} \theta_j \theta_{j+\tau} \right) \sigma_\varepsilon^2$$

- 3. Spectral representation :

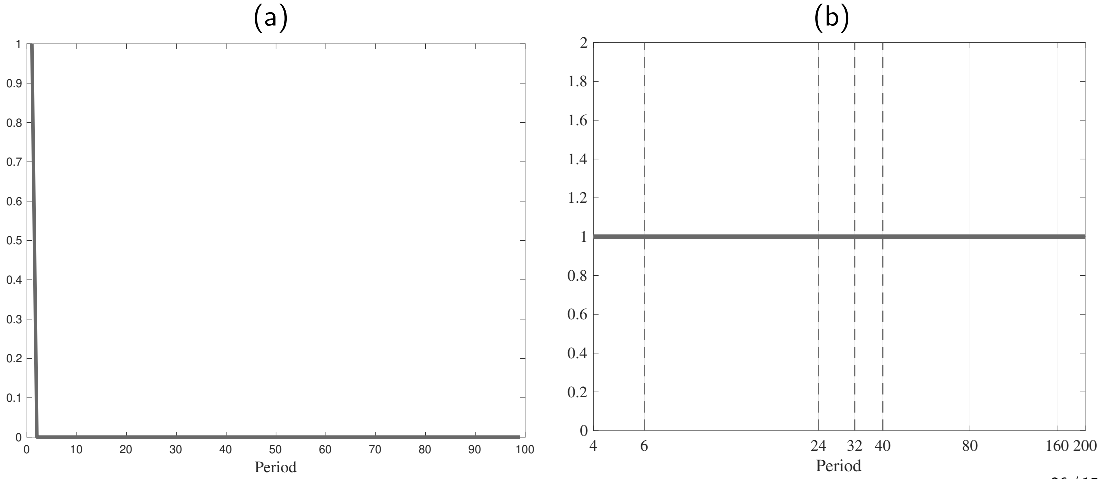
$$s(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{+\infty} \lambda_\tau (\cos(\omega\tau) + i \sin(\omega\tau))$$

- Note : works only for stationary series

I. Cyclicality

$x_t = \varepsilon_t$

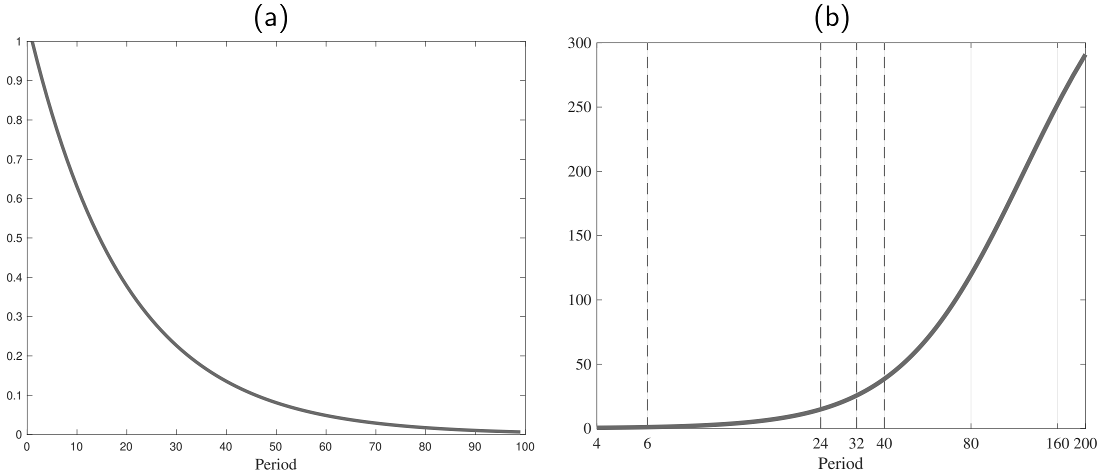
FIGURE 5 – (a) IRF and (b) Spectrum



I. Cyclicality

$x_t = .95x_{t-1} + \varepsilon_t$

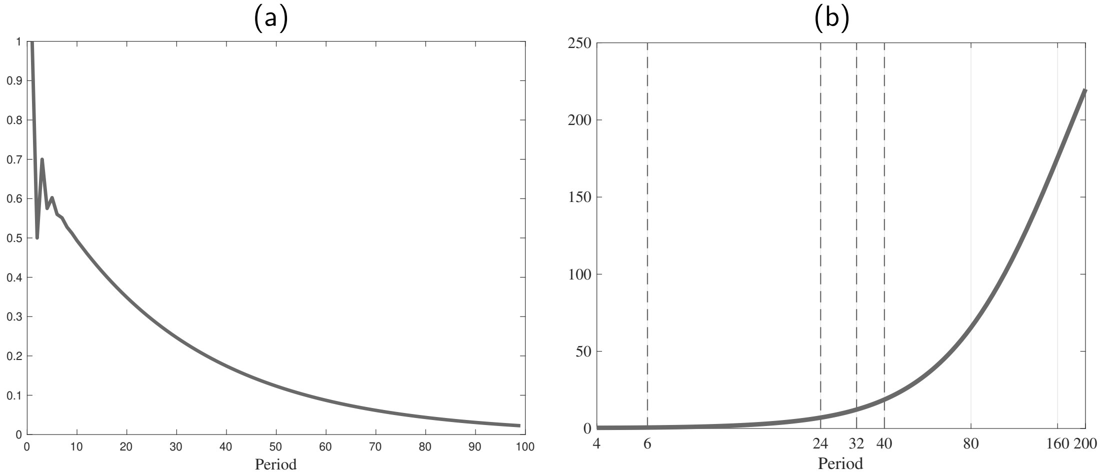
FIGURE 6 – (a) IRF and (b) Spectrum



I. Cyclicality

$x_t = .5x_{t-1} + .45x_{t-2} + \varepsilon_t$

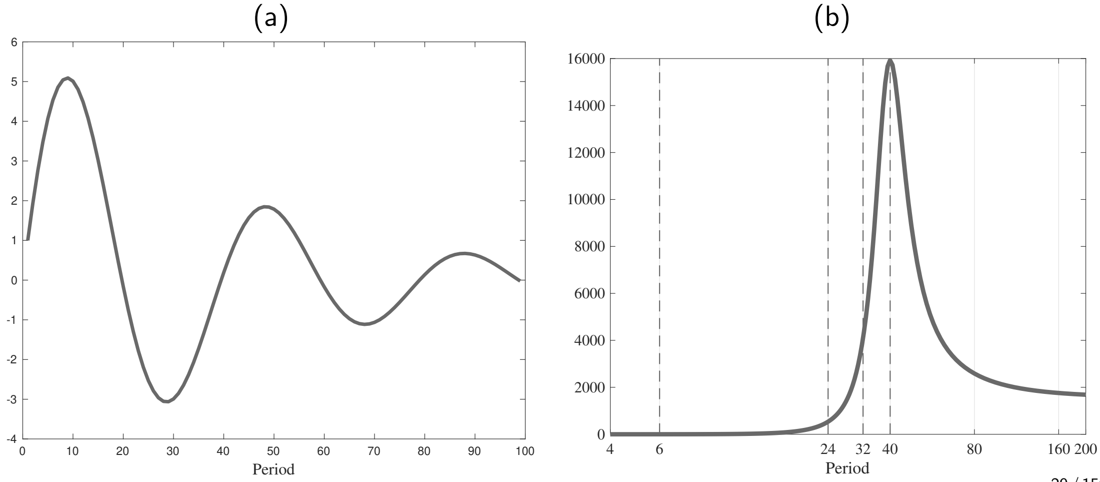
FIGURE 7 – (a) IRF and (b) Spectrum



I. Cyclicality

$x_t = 1.92x_{t-1} - .95x_{t-2} + \varepsilon_t$

FIGURE 8 – (a) IRF and (b) Spectrum



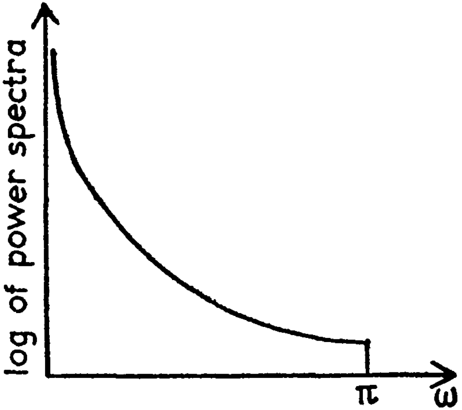


FIGURE 1.—Typical spectral shape.

I. Cyclicality

Conventional Wisdom-GRANGER [1969]

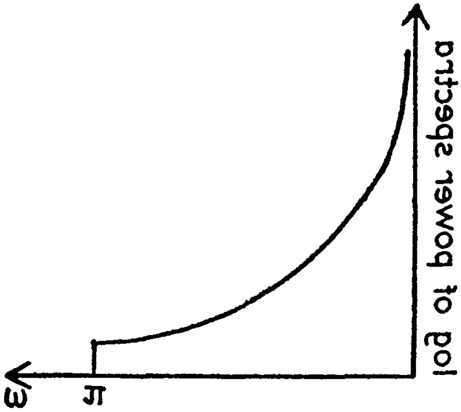
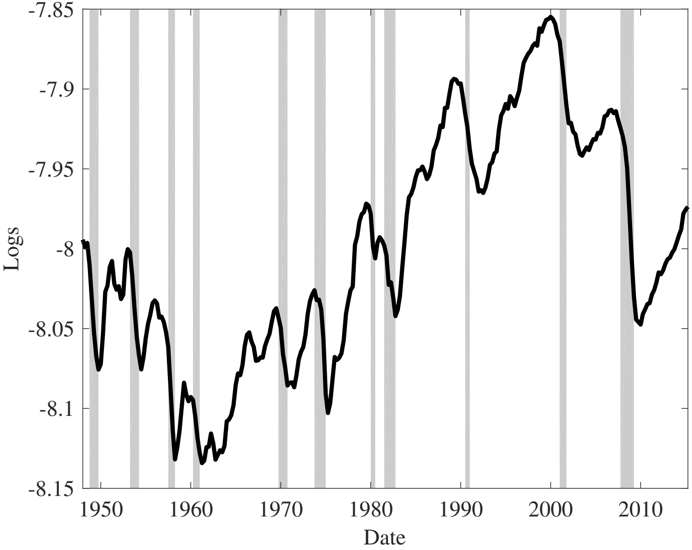


FIGURE 1.—Typical spectral shape.

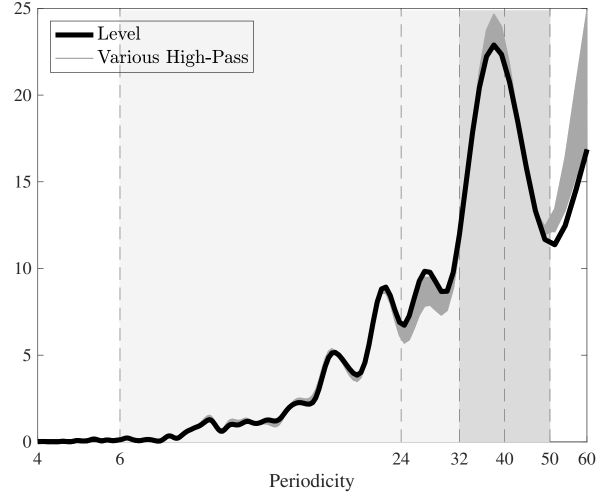
I. Cyclicality

Non-Farm Business (NFB) Hours Per Capita



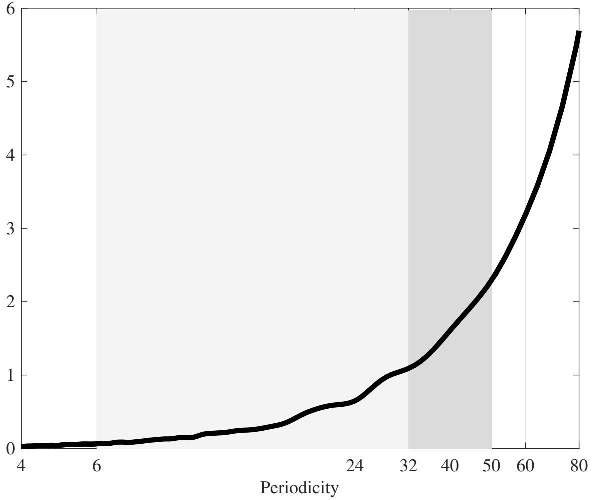
I. Cyclicality

Non Farm Business Hours per Capita Spectrum



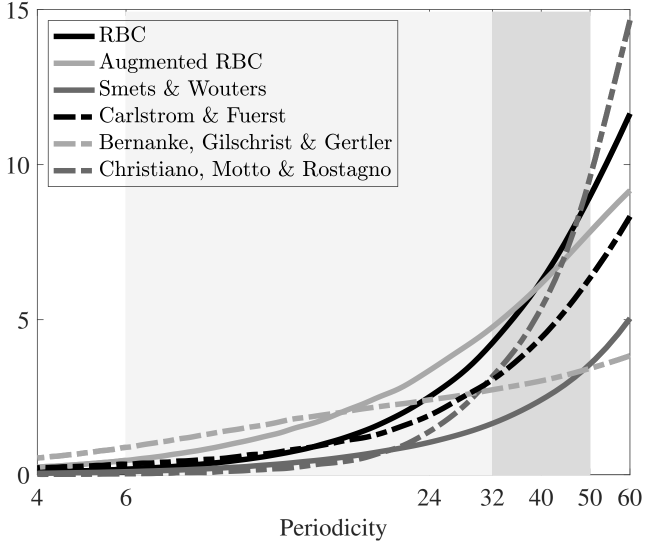
I. Cyclicality

Hours Spectrum in Smets & Wouters' Model



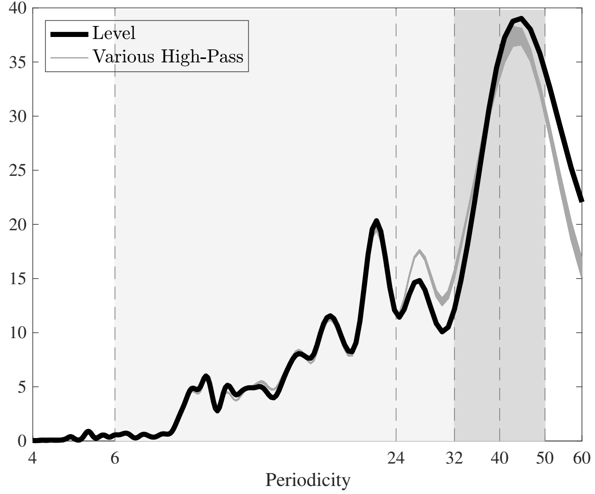
I. Cyclicality

Hours Spectrum in Various Models



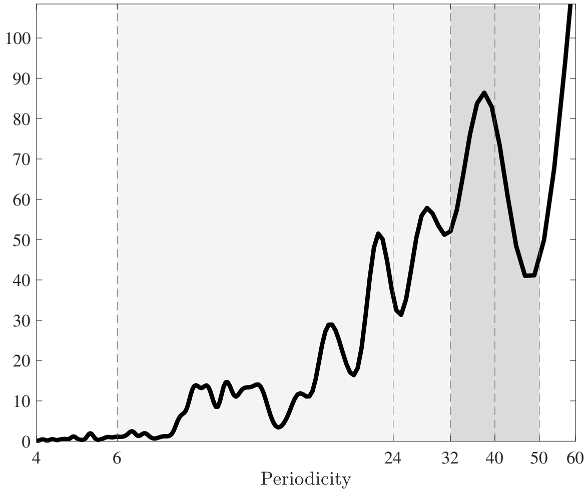
I. Cyclicality

Capacity Utilization Spectrum



I. Cyclicality

Investment-Output ratio

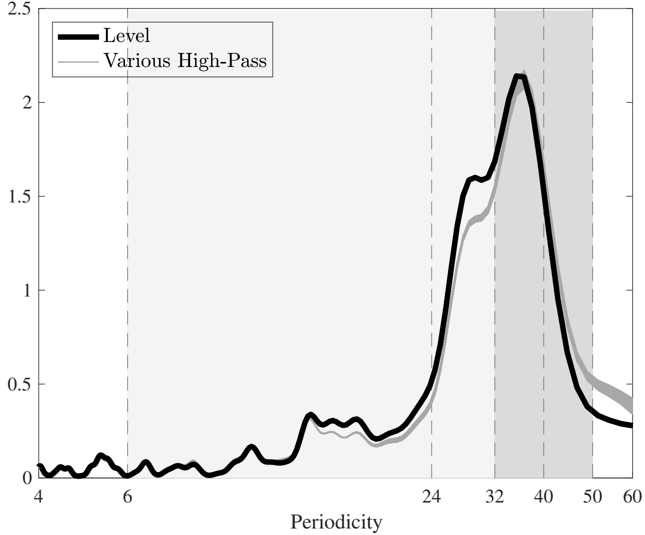


I. Cyclicality

- The cycle is also a financial cycle
- (looking again at non-trending variables)

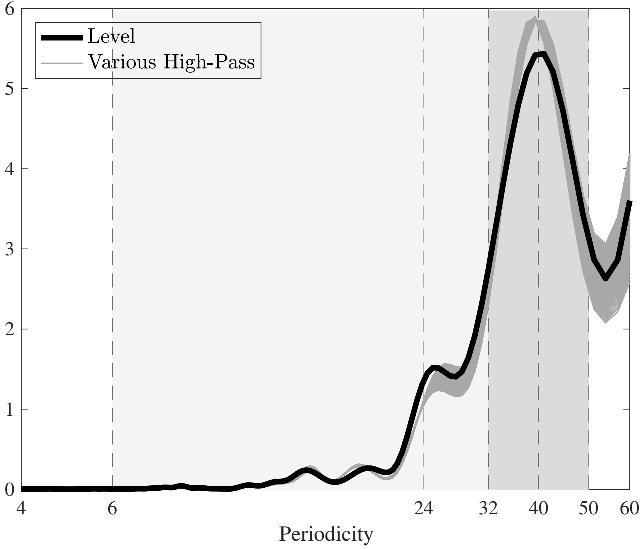
I. Cyclicality

Chicago Fed National Financial Conditions Index



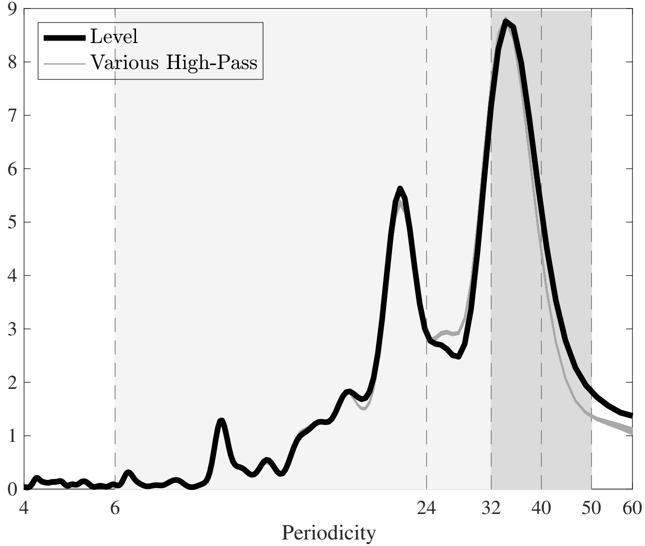
I. Cyclicality

Delinquency Rate



I. Cyclicality

Spread (BBA bonds-FFR)



- Traditional view of business cycles is a-cyclical :
  - × Spectral densities are thought monotonous
  - × This is what most models endogenously produce
  - × Exogenous forces do not look cyclical
- Data seems to tell us that there are indeed cycles
  - × because we have more observations
  - × because we look at non-trending variables (no need for stationarization)

- Let X_t be a cyclical measure of activity, example : hours worked or unemployment rate.
- Consider estimating an AR process (assuming zero mean).

$$X_t = A(L)X_{t-1} + \epsilon_t$$

  - × If roots of $I - A(L)$ are sufficiently outside of unit circle, we tend to take as evidence of stability.
  - × If very close to 1, we worry about a unit root.
- We often tend to disregard the possibility of roots inside the unit disc,
  - × as this would imply explosive behavior ;
  - × because it is not found in the data

- But suppose instead the DGP process is of form

$$X_t = A(L)X_{t-1} + \gamma F(X_{t-1}) + \epsilon_t \qquad F(0) = F'(0) = 0$$

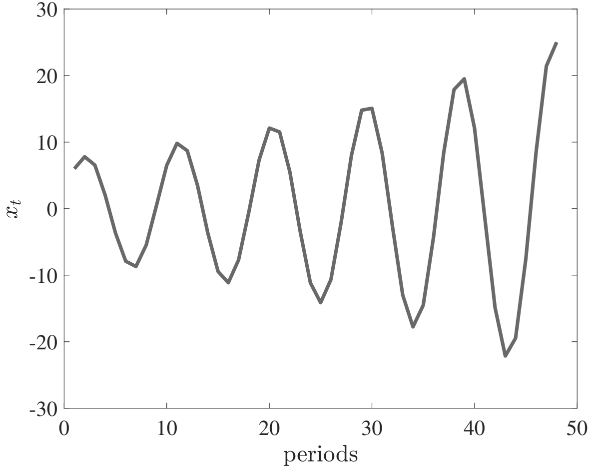
- where $F(\cdot)$ is a non-linear function and γ may be very small.
- The stability of the zero steady state will still depend only on roots of $I - A(L)$.
- Hence, linear approx unchanged and dropping $\gamma F(X_{t-1})$ in estimation may seem reasonable.
- However, this could lead to substantial bias in estimation of roots of $A(L)$ if system is locally unstable.
- Consider AR(3) example with small cube term ($F(x) = x^3$).

- Assume the following DGP

$$x_t = \alpha x_{t-1} - 0.6x_{t-2} - 0.3x_{t-3} - 0.01x_{t-1}^3 + .25\epsilon_t, \qquad (2)$$

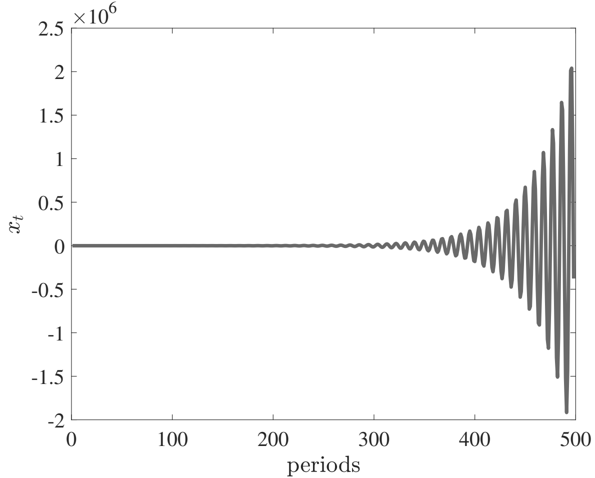
- α takes values in $[0.5, 1.5]$
- $\rightsquigarrow |\lambda|_{\text{max}} \in [0.94, 1.4]$.
- Let's look at the case in which $|\lambda|_{\text{max}} = 1.026$.

Figure 9 – Theoretical Impulse Response when ν is set to 0



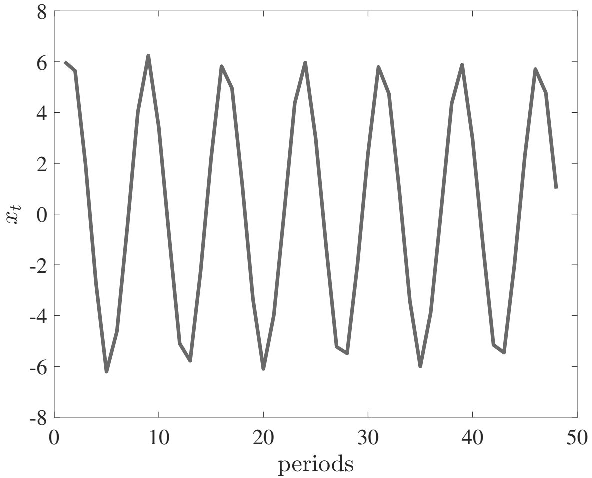
$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

FIGURE 10 – Theoretical Impulse Response when ν is set to 0



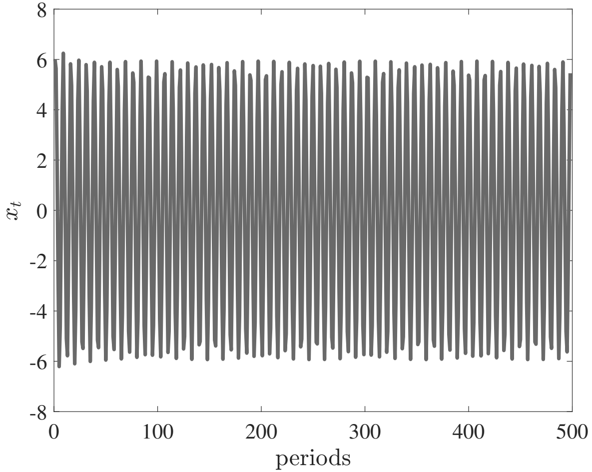
$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

FIGURE 11 – Theoretical Impulse Response in the Nonlinear Model (ν negative)



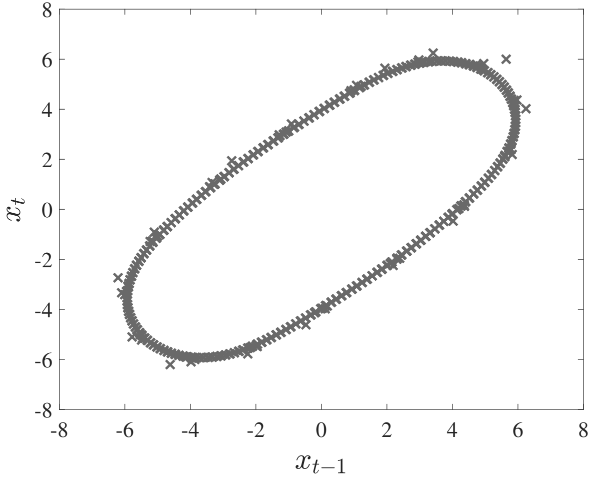
$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

Figure 12 – Theoretical Impulse Response in the Nonlinear Model



$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

FIGURE 13 – The Limit Cycle in the Nonlinear Model



$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

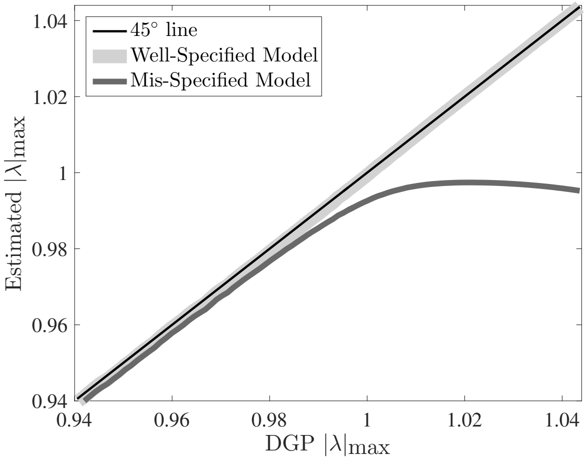
$$x_t = \alpha x_{t-1} - 0.6x_{t-2} - 0.3x_{t-3} - 0.01x_{t-1}^3 + .25\epsilon_t, \tag{3}$$

- Take many α in $[0.5, 1.5]$
- $\rightsquigarrow |\lambda|_{\max} \in [0.94, 1.4]$.
- Simulated the DGP (3) for each α and estimate the two specifications :

$$\begin{aligned} x_t &= \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} \quad + \nu x_{t-1}^3 \quad + \epsilon_t \\ x_t &= \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} \qquad\qquad\quad + \epsilon_t \end{aligned}$$

- Compare estimated $|\lambda|_{\max}$ for well- and mis-specified models with the true $|\lambda|_{\max}$.

Figure 14 – $|\lambda|_{\max}$ for the Linear and Nonlinear estimation when the DGP is Nonlinear



II. Instability

Introductory example

FIGURE 15 – R^2 for the Linear and Nonlinear estimation when true $|\lambda|_{\text{max}} = 1.02$ (1000 simulations, 300 observations per simulation)

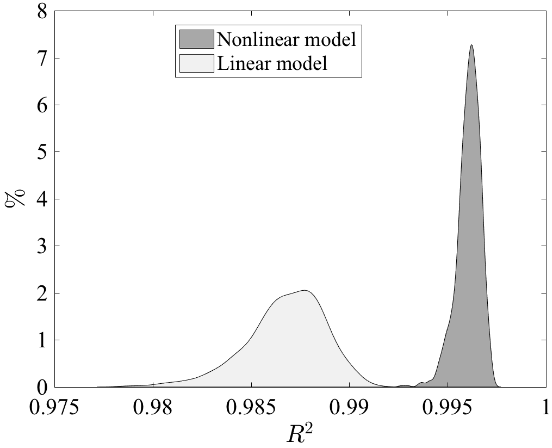
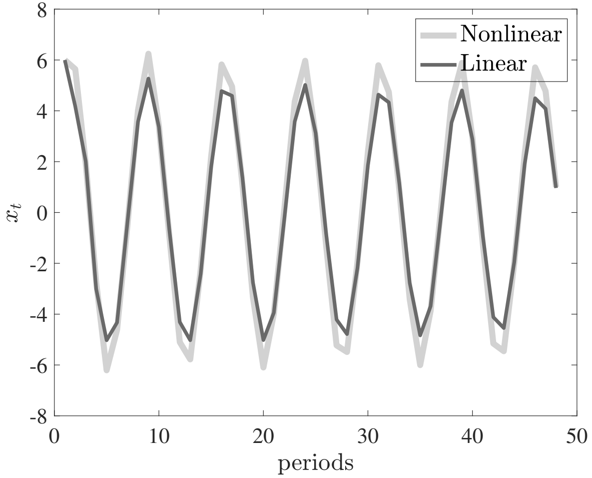
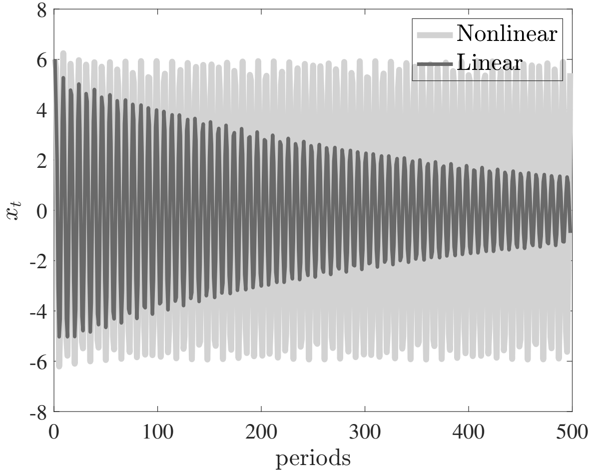


FIGURE 16 – Estimated Nonlinear and Linear Model Impulse Response



$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

FIGURE 17 – Estimated Nonlinear and Linear Model Impulse Response



$$|\lambda|_{\max} = 1.026, \quad x_t = \alpha x_{t-1} + \beta x_{t-2} + \gamma x_{t-3} + \nu x_{t-1}^3 + \epsilon_t$$

- This suggests a specific way of looking at the data
- h : Total Hours Worked per Capita, U.S.A., 1960-2015
- High-Pass Filtered, 80 quarters
- "Minimal" model

$$\begin{cases} h_t &= \alpha_0 + \alpha_1 h_{t-1} + \alpha_2 h_{t-2} + \alpha_3 H_{t-1} + \alpha_4 h_{t-1}^3 + \varepsilon_t \\ H_t &= \sum_{j=0}^{N} (1-\delta)^j h_{t-j} \end{cases}$$

II. Instability

Estimated Reduced Form

$$\begin{cases} h_t &= -0.00 + 1.42\ h_{t-1} - 0.48\ h_{t-2}, \\ h_t &= -0.01 + 1.31\ h_{t-1} - 0.34\ h_{t-2} - 0.25\ H_{t-1}, \\ h_t &= -0.02 + 1.39\ h_{t-1} - 0.34\ h_{t-2} - 0.27\ H_{t-1} - 0.01\ h_{t-1}^3. \end{cases}$$

- Non-linear term is significant
- Non-linear term enters with a *minus*
-

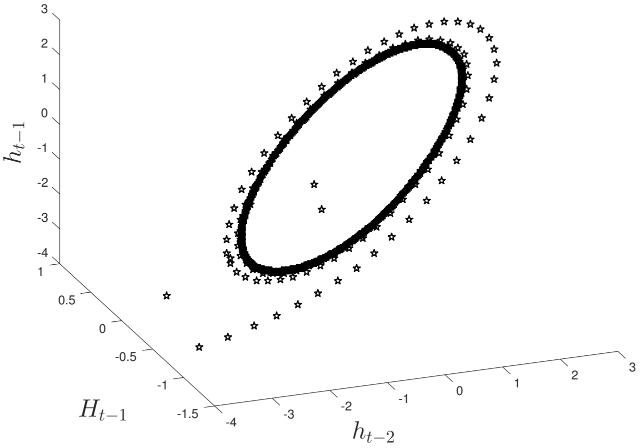
	$AR(2)$	Linear	Minimal
R^2	0.94	0.94	0.94
Max eig.	0.86	0.96	1.01

- R^2 is not much improved
- But max eigenvalue (in modulus) crosses 1 with the nonlinear term
- SS is unique, unstable

II. Instability

Estimated Reduced Form - Total Hours

FIGURE 18 – The Limit Cycle - Simulation as of $T_0 = 1961$



Estimated Reduced Form - Total Hours

FIGURE 19 – The Limit Cycle - Simulation as of $T_0 = 1961$

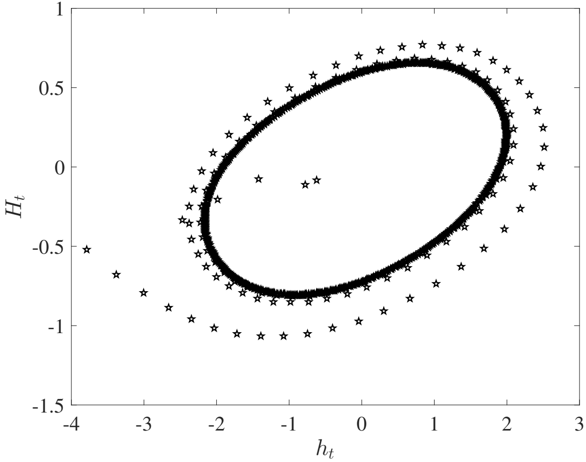
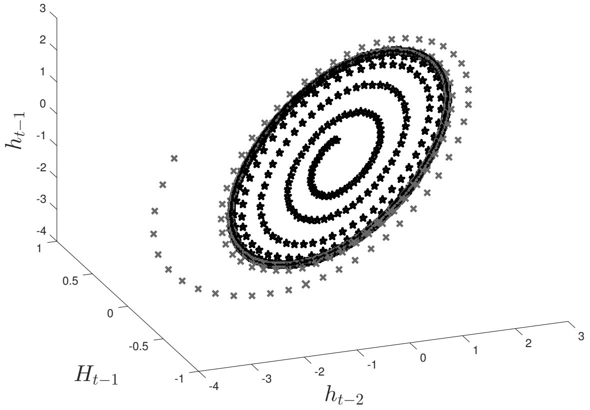


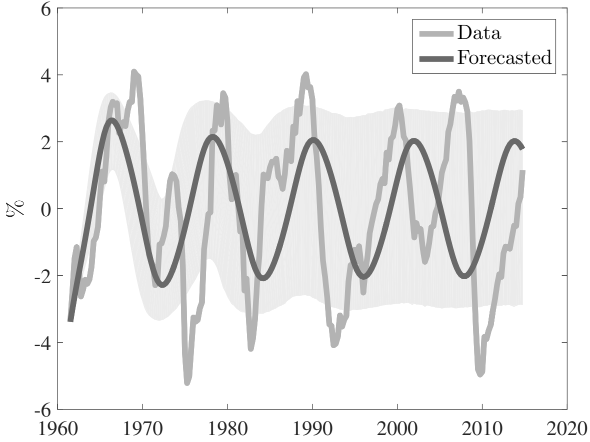
FIGURE 20 – The Limit Cycle



II. Instability

Estimated Reduced Form - Total Hours

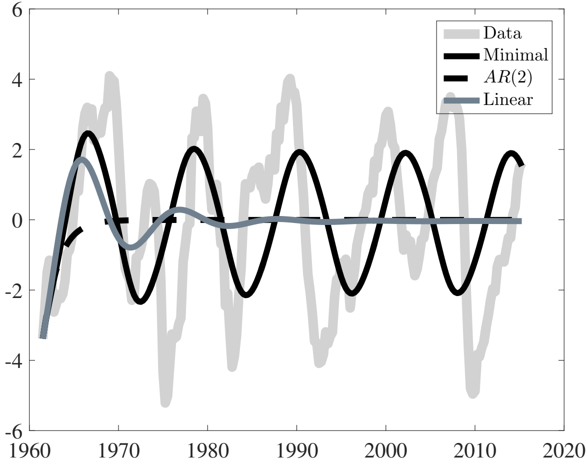
FIGURE 21 – Forecasted Path as of 1961Q3 with the Minimal Model, Total Hours



II. Instability

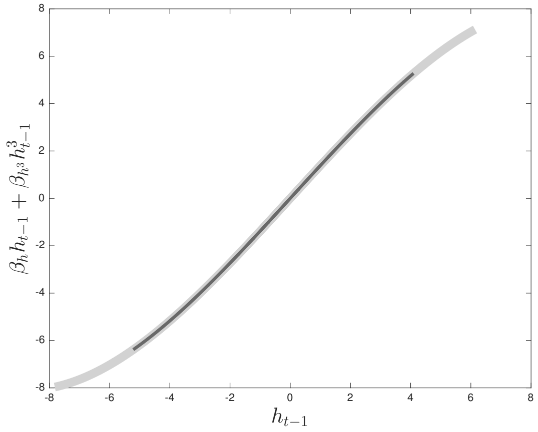
Estimated Reduced Form - Total Hours

Figure 22 – Forecasted Path as of 1961Q3, Total Hours



Estimated Reduced Form - Total Hours

FIGURE 23 – Nonlinearities in the Minimal Model, Total Hours



$$h_t = -0.02 + 1.39\ h_{t-1} - .34\ h_{t-2} - .27 H_t - .01\ h_{t-1}^3 + \epsilon_t$$

Roadmap

I. Cyclicality

II. Instability

III. Generating Cycles through Dynamic Models with Interactions

IV. A Fully Specified Model

III. Generating Cycles through Dynamic Models with Interactions

- Which theoretical structure can produce cycles and instability ?
- Individual behaviours are not cyclical
- Robinson Crusoe is unlikely to see booms and busts other than those caused by nature on his island
- But if we put one million of Crusoes with same preferences and technology together, booms and busts more likely
- Cycles as an emergent phenomenon

- Activity Y_t depends positively on balance sheet conditions of HH or firms
- Balance sheet conditions X_t depend positively on activity

$$
\begin{aligned}
Y_t &= \alpha_1 Y_{t-1} + \alpha_2 X_t + \epsilon_t \\
X_t &= \alpha_3 X_{t-1} + \alpha_4 Y_t + \mu_t
\end{aligned}
$$

- All parameters are positive.

Proposition 1

The class of endogenous mechanism <u>cannot</u> create a hump shaped spectral density.

- Question : What forces are "necessary", assuming individual level behavior is not itself cyclical ?

III. Generating Cycles through Dynamic Models with Interactions

Environment (Might be thought as an Agent Based Model)

- N players
- Each agent accumulates X_i by playing e_i
- Decision rule and law of motion for X are

$$X_{it+1} = (1-\delta)X_{it} + e_{it} \tag{4}$$

$$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F(e_t) \tag{5}$$

  with $e_t = \frac{\sum e_{jt}}{N}$

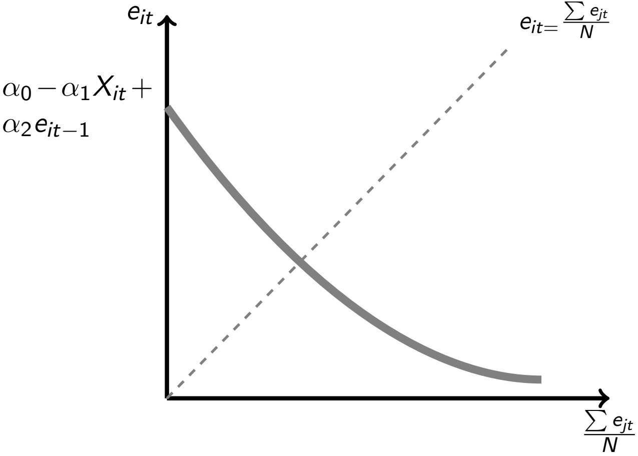
- $\delta < 1$, $0 < \alpha_2 < 1$.
- $\alpha_1 X_{it}$: Optimal size argument (if $\alpha_1 > 0$)
- $\alpha_2 e_{it-1}$: Adjustment cost argument
- $F(e_t)$: Strategic Interactions (a price, a quantity constraint)
  - × $F' > 0$: strategic complementarities
  - × $F' < 0$: strategic substitutabilities

$$
\begin{aligned}
X_{it+1} &= (1-\delta)X_{it} + e_{it} \\
e_{it} &= \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F\left(e_t\right)
\end{aligned}
$$

- Define (X^s, e^s) as the steady state of the linear model where $F(\cdot) = 0$
- Normalize $F(e^s) = 0$

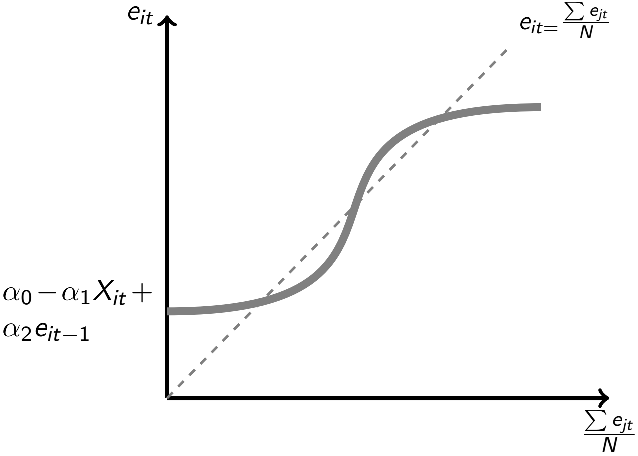
III. Generating Cycles through Dynamic Models with Interactions

"Best response" rule for a given history - Strategic substitutability



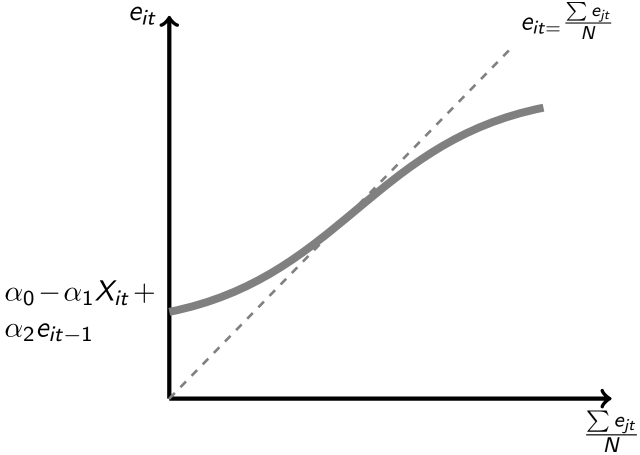
$$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{jt}}{N}\right)$$

"Best response" rule for a given history - Multiple Equilibria



$$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{jt}}{N}\right)$$

- We restrict to the case where $F'(\cdot) < 1$, so that there are never multiple equlibria *within* period t



2. Abstract Framework

A Proposition

$$
\begin{aligned}
e_{it} &= \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F\left(e_t\right) \\
X_{it+1} &= (1-\delta) X_{it} + e_{it}
\end{aligned}
$$

- Aggregate outcome satisfies (locally)

$$
\begin{aligned}
e_t &= \alpha_0 - \alpha_1 X_t + \alpha_2 e_{t-1} + F'(e^s)\ e_t \\
X_{t+1} &= (1-\delta) X_t + e_t
\end{aligned}
$$

Proposition 2

Necessary condition for this system to produced hump shaped spectral density : complementarities ($F' > 0$) and dampening effect of stock ($\alpha_1 > 0$)

$$
\begin{aligned}
e_t &= \alpha_0 - \alpha_1 X_t + \alpha_2 e_{t-1} + F'(e^s)\ e_t \\
X_{t+1} &= (1-\delta) X_t + e_t
\end{aligned}
$$

- Even if $F'(e^s) < 1$, this system will become unstable if complementarity sufficiently strong. (although non indeterminacy)
- Moreover, the loss of stability will generally happen when roots are complex.
- So this is a system where one should be aware that the presence of non-linearities – for example in the interaction function $F(e_t)$ – may cause sustained cycles : limit cycles.
- One should not rule out the (local) instability of the SS in the estimation.
- The potential presence of (stochastic) limit cycles causes new challenges for estimation as the steady state become unstable.
- With forward looking elements, this give rise to notion of saddle path stable limit cycles

- Assume $\alpha_2 = 0$ in

$$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{jt}}{N}\right)$$

- Then the model is

$$\begin{aligned} X_{it+1} &= (1-\delta)X_{it} + e_{it} \\ e_{it} &= \alpha_0 - \alpha_1 X_{it} + F(e_t) \end{aligned}$$

- It boils down to one order one dynamic equation in X_{it}

$$X_{it+1} = \alpha_0 + (1 - \delta + \alpha_1)X_{it} + F(X_{t+1} - (1-\delta)X_t)$$

- So that, for symmetrical allocations

$$X_{t+1} = \alpha_0 + (1 - \delta + \alpha_1)X_t + F(X_{t+1} - (1-\delta)X_t)$$

- Assume no strategic interactions : $F(\cdot) \equiv 0$

$$X_{t+1} = \alpha_0 + (1 - \delta - \alpha_1)X_t$$

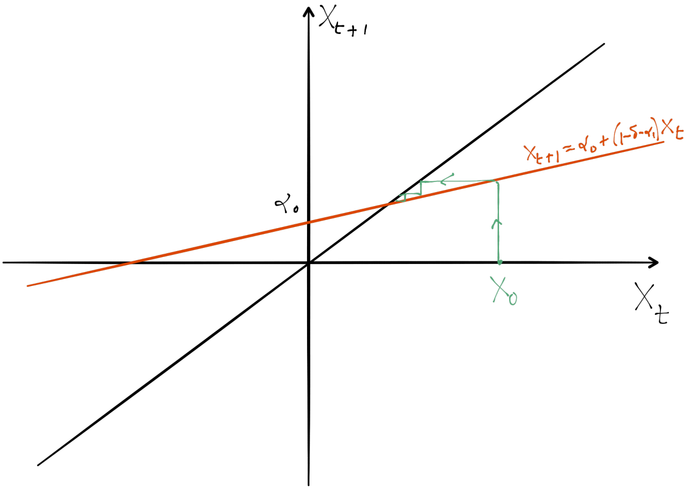
- Note that $(1 - \delta - \alpha_1)$ can be either > 0 or < 0.

Proposition 3

The Economy is globally stable as $|1 - \delta - \alpha_1| < 1$

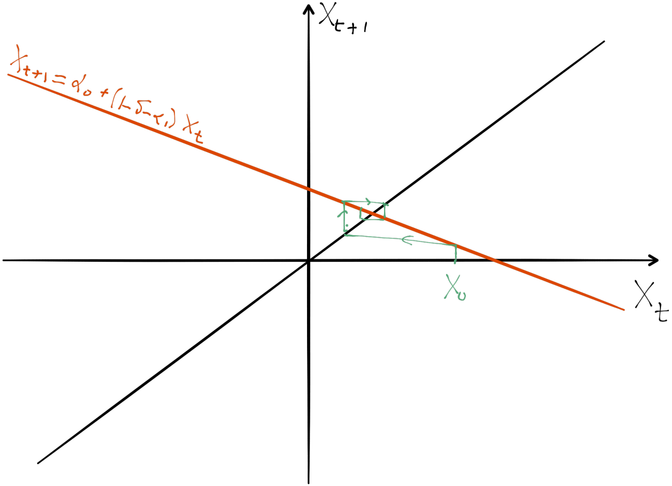
III. Generating Cycles through Dynamic Models with Interactions

One-Dimension Case ($\alpha_1 < 0$, no cycles)



III. Generating Cycles through Dynamic Models with Interactions

One-Dimension Case ($\alpha_1 > 0$, cycles)



- Reintroduce strategic interactions

$$X_{t+1} = \alpha_0 + (1 - \delta + \alpha_1)X_t + F(X_{t+1} - (1 - \delta)X_t)$$

- Linearize around the steady state X^s

$$X_{t+1} = (\alpha_0 + \delta X^s) + (1 - \delta + \alpha_1)X_t + F'(e^s)(X_{t+1} - (1 - \delta)X_t)$$

- so that

$$X_{t+1} = \frac{\alpha_0 + \delta X^s}{1 - F'(e^s)} + \frac{((1 - \delta)(1 - F'(e^s)) + \alpha_1)}{1 - F'(e^s)} \; X_t$$

$$X_{t+1} = \frac{\alpha_0 + \delta X^s}{1 - F'(e^s)} + \frac{((1-\delta)(1-F'(e^s)) + \alpha_1)}{1 - F'(e^s)} \quad X_t$$

Proposition 4

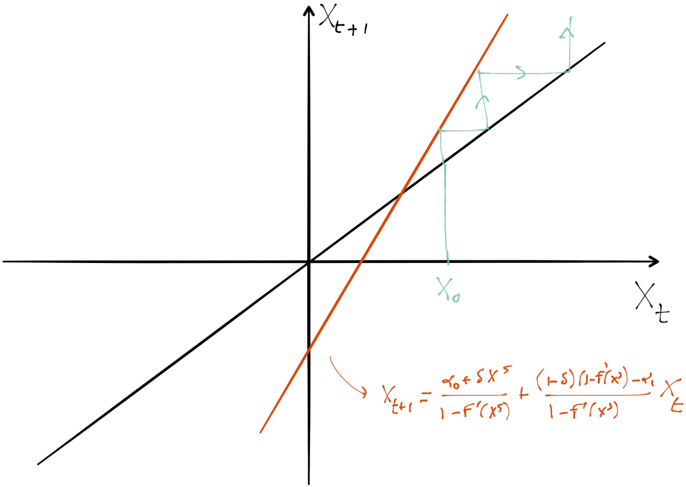
With strategic substitutability ($F'(\cdot) < 0$), the economy is always stable

Proposition 5

With strategic complementarity ($F'(\cdot) > 0$), there is always a level of complementarity smaller than 1 such that the SS is (locally) unstable.

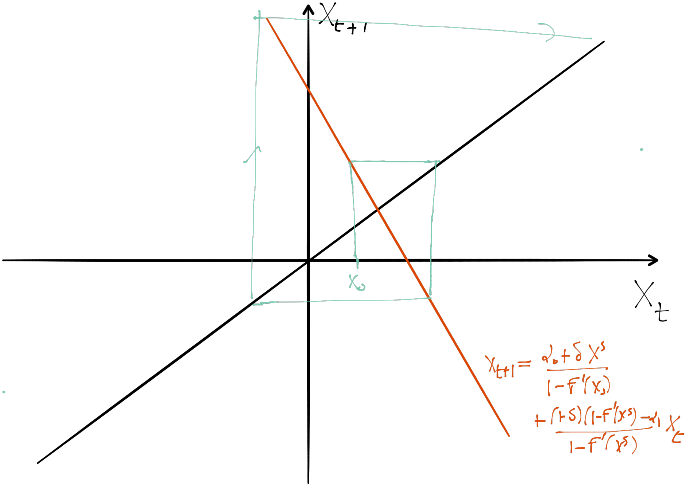
III. Generating Cycles through Dynamic Models with Interactions

One-Dimension Case ($\alpha_1 < 0$, no cycles)

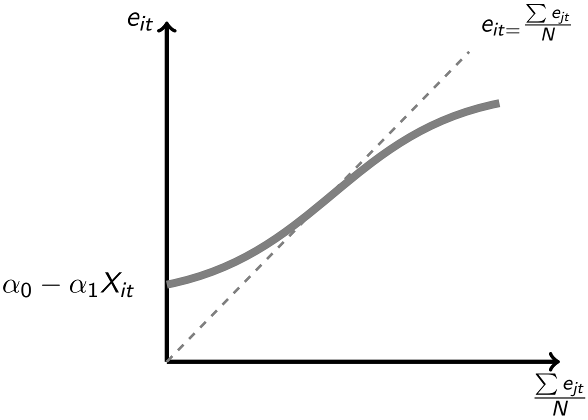


III. Generating Cycles through Dynamic Models with Interactions

One-Dimension Case ($\alpha_1 > 0$, cycles)



- The SS X^s becomes locally unstable
- "Real data" do not look explosive
- Let's assume that strategic complementarities dies out when the economy is far above of below X^s
- F is "S-shaped"



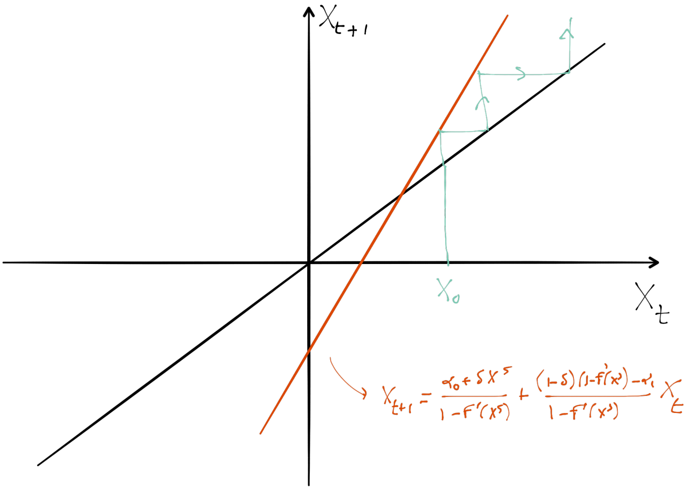
- Assume for simplicity that F is symmetric wrt X^s

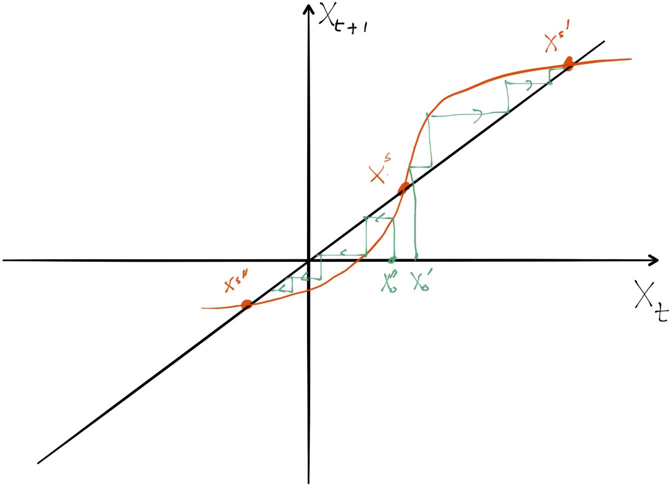
$$X_{t+1} = \frac{\alpha_0 + \delta X^s}{1 - F'(e^s)} + \frac{((1-\delta)(1-F'(e^s)) + \alpha_1)}{1 - F'(e^s)} \; X_t$$

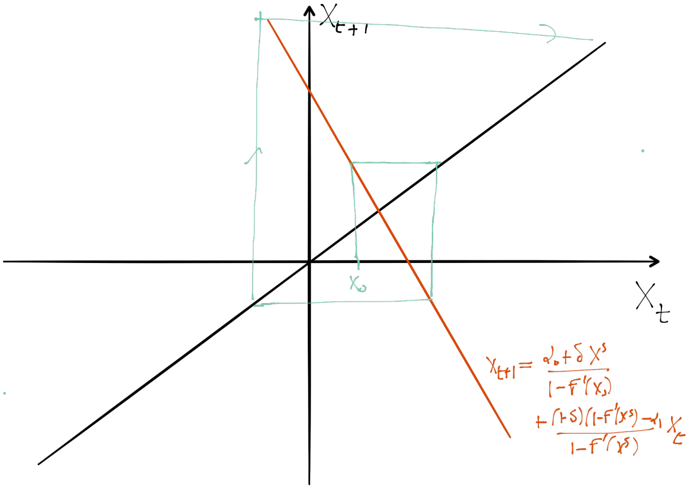
- The dynamics of the economy will very much depend on whether $\frac{((1-\delta)(1-F'(e^s))+\alpha_1)}{1-F'(e^s)}$ is positive or negative
  - × Positive : Hysteresis
  - × Negative : Limit cycle
- Need to think of environments that makes this slope is positive or negative

III. Generating Cycles through Dynamic Models with Interactions

One-Dimension Case : Adding Nonlinearities



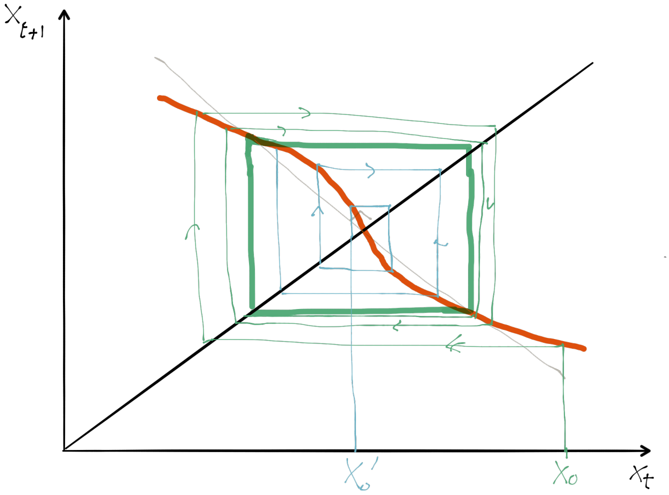

X_{t+1}
X_t
$X^{s'}$
X^s
$X^{s''}$
X_0''
X_0'



X_{t+1}
X_t
X_0
$X_{t+1} = \frac{\alpha_0 + \delta X^s}{1 - F'(X_s)} + \frac{(1-\delta)(1-F'(X^s)) - \alpha_1}{1 - F'(X^s)} X_t$

III. Generating Cycles through Dynamic Models with Interactions

One-Dimension Case : Adding Nonlinearities



- Problem with one dimension model : the dynamics does not look like a business cycle
- X_t is *not* positively correlated.
- This needs not to be true in the more general model where $\alpha_2 > 0$

$$
\begin{aligned}
X_{it+1} &= (1-\delta)X_{it} + e_{it} \\
e_{it} &= \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F(e_t)
\end{aligned}
$$

- Local dynamics is

$$
\begin{pmatrix} e_t \\ X_t \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\alpha_2-\alpha_1}{1-F'(e^s)} & -\frac{\alpha_1(1-\delta)}{1-F'(e^s)} \\ 1 & 1-\delta \end{pmatrix}}_{M} \begin{pmatrix} e_{t-1} \\ X_{t-1} \end{pmatrix}
+ \begin{pmatrix} \left(1-\frac{\alpha_2-\alpha_1}{1-F'(e^s)}\right) e^s - \left(\frac{\alpha_1(1-\delta)}{1-F'(e^s)}\right) X^s \\ 0 \end{pmatrix}
$$

- When $F'(e^s)$ varies from $-\infty$ to 1, eigenvalues of M vary

Proposition 6

As $F'(e^s)$ varies from 0 to $-\infty$, the eigenvalues of M always stay within the unit circle and therefore the system remains locally stable.

Proposition 7

As $F'(e^s)$ varies from 0 towards 1, the dynamic system will become locally unstable.

(bifurcation)

- 3 types of bifurcation
  - Fold bifurcation : appearance of an eigenvalue equal to 1,
  - Flip bifurcation : appearance of an eigenvalue equal to -1
  - HOPF bifurcation : appearance of two complex conjugate eigenvalues of modulus 1
    $\rightsquigarrow$ hump is spectral density

- We are interested in HOPF bifurcation because the limit cycle will be "persistent"

Proposition 8

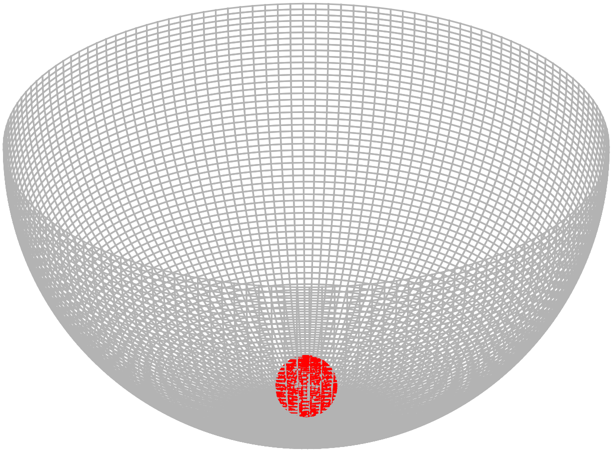
As $F'(e^s)$ varies from 0 towards 1, the dynamic system will become unstable and we'll have smooth cycles (HOPF) if $\alpha_1 > 0$ and α_2 is large enough.

$$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F(e_t)$$

- Discrete time version of the HOPF theorem.
- Nice theorem : we simply have to look at the *linearized* dynamics to prove existence of a limit cycle
- The parameter that varies is here the degree of strategic complementarities at the steady state $F'(e^s)$
- It is quite intuitive why a limit cycle occurs when the steady state moves from stable to unstable

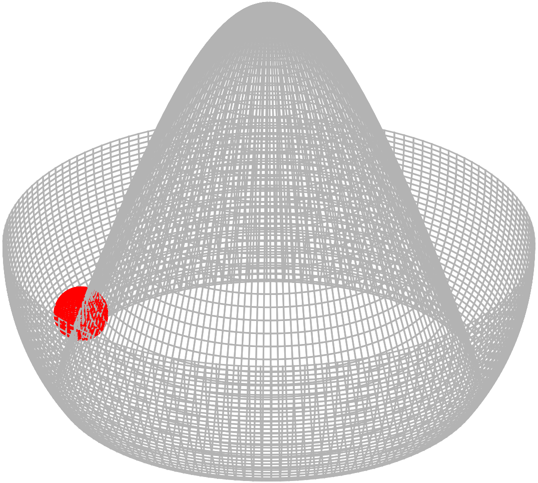
- Here we can have a limit cycle with persistence
- Consider the steady state (X^s, e^s)
- Strategic complementarities : centrifugal force that pushes away from the steady state when close to.
- Accumulated variable X : centripetal force that pushes towards the steady state when away from.
- The steady state locally unstable, but forces push the economy back to the steady state when it is further from it.
- It is quite intuitive why a limit cycle occurs when the steady state moves from stable to unstable
- In the case of the HOPF bifurcation, the limit cycle can be attractive (*the bifurcation is supercritical*) or repulsive (*the bifurcation is subcritical*)

FIGURE 24 – Stable Steady State



III. Generating Cycles through Dynamic Models with Interactions

FIGURE 25 – HOPF Supercritical bifurcation : Attractive Limit Cycle



III. Generating Cycles through Dynamic Models with Interactions

Figure 26 – Unstable Steady State

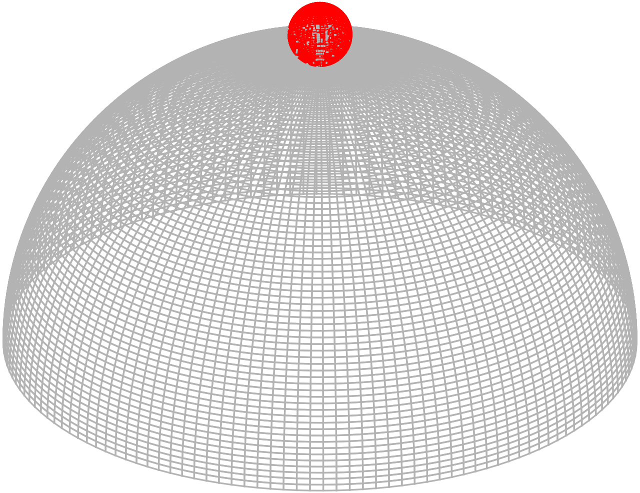
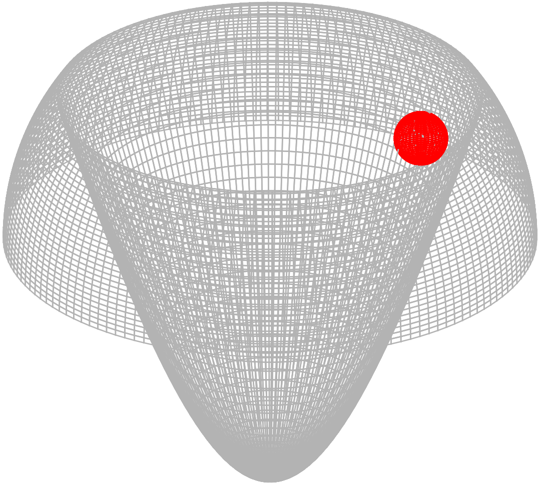
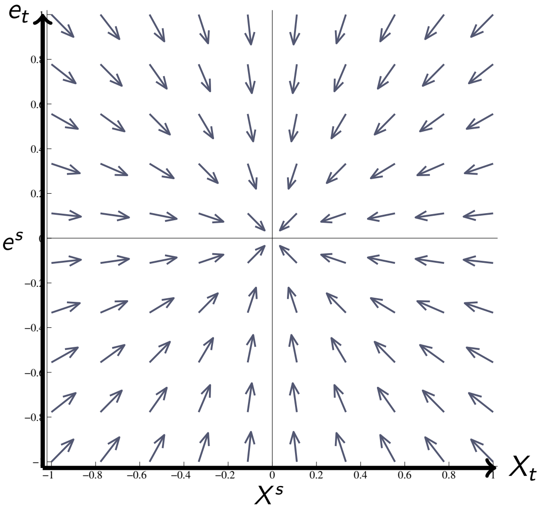


Figure 27 – Hopf Subcritical bifurcation : Repulsive Limit Cycle

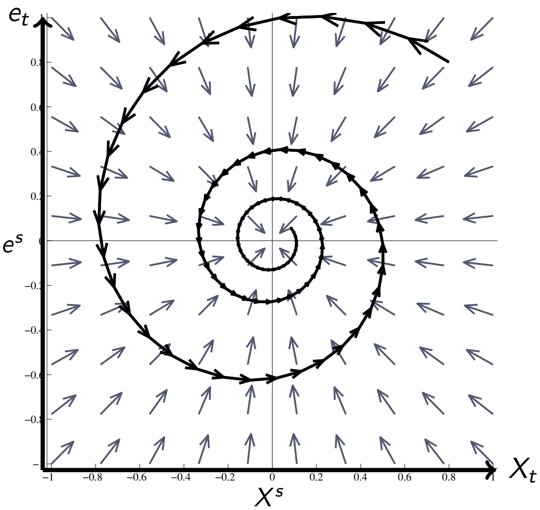


Two-Dimension Case - Global stability

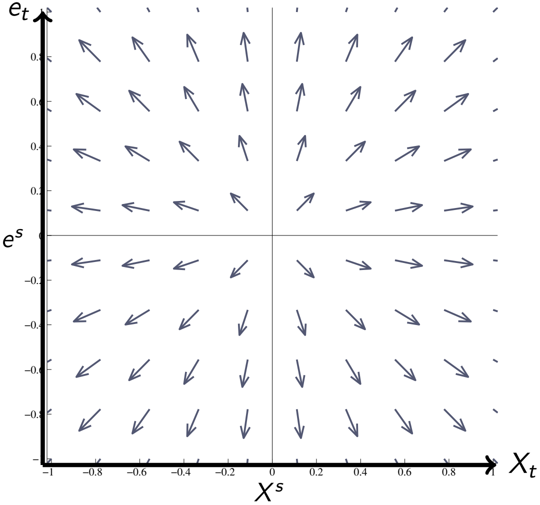


III. Generating Cycles through Dynamic Models with Interactions

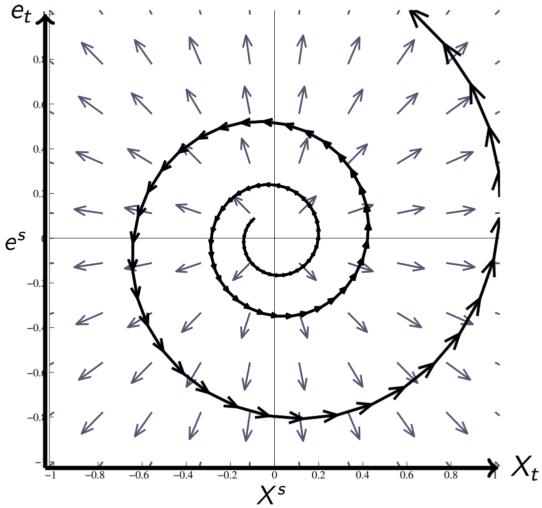
Two-Dimension Case - Global stability



Two-Dimension Case - Global instability

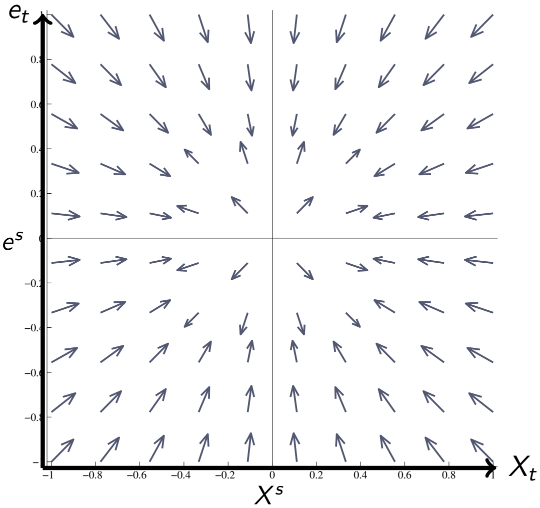


Two-Dimension Case - Global instability



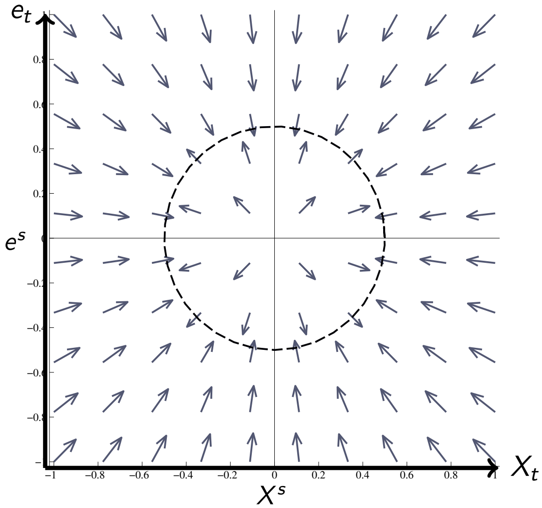
III. Generating Cycles through Dynamic Models with Interactions

Two-Dimension Case - Stable limit cycle

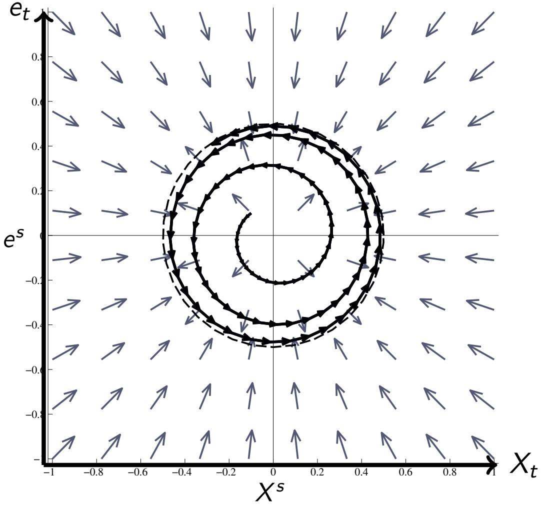


III. Generating Cycles through Dynamic Models with Interactions

Two-Dimension Case - Stable limit cycle

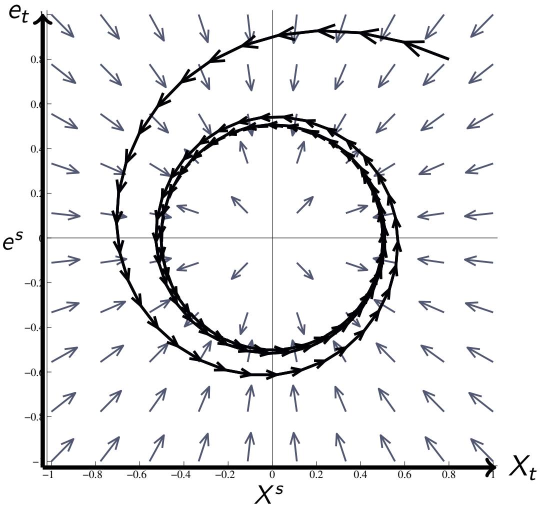


Two-Dimension Case - Stable limit cycle



III. Generating Cycles through Dynamic Models with Interactions

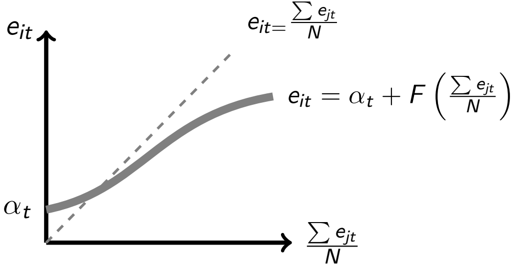
Two-Dimension Case - Stable limit cycle



Proposition 9

If $F'''(e^s)$ is sufficiency negative, then the HOPF *bifurcation will be supercritical. Therefore, the limit cycle is attractive.*

-
- $F''' < 0$ corresponds to an $S-$ *shaped* reaction function

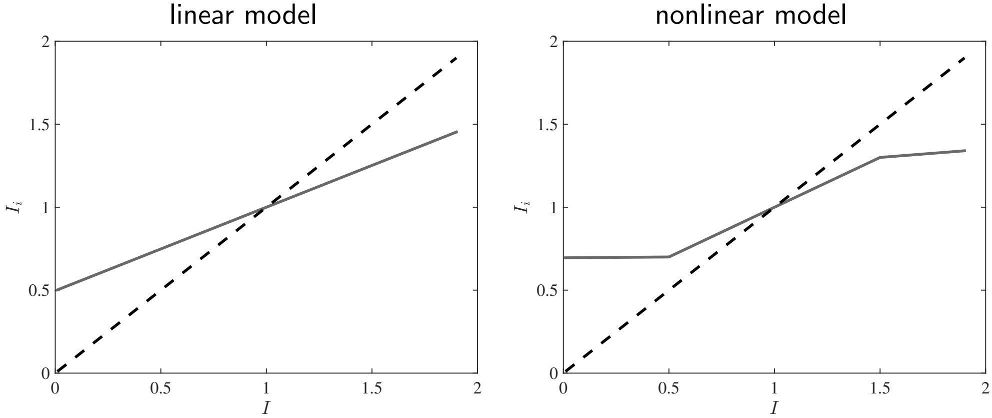


-

$$\alpha_t = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1}$$

- $X_{it+1} = (1-\delta)X_{it} + e_{it}$
- $e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + F\left(\frac{\sum e_{it}^s}{N}, u_t\right)$
- $F(e_t, u_t) = \widetilde{F}(e_t) + u_t.$
- $u_t = \rho u_{t-1} + \varepsilon_t$
- $\widetilde{F}$: S-shaped and piecewise linear

FIGURE 28 – Best response rules in the numerical example



- Steady state is $e_{SS} = 1$, $X_{SS} = 10$
- Deterministic simulation : let $X_0 = 8$, $e_0 = 1$

FIGURE 29 – Deterministic simulation

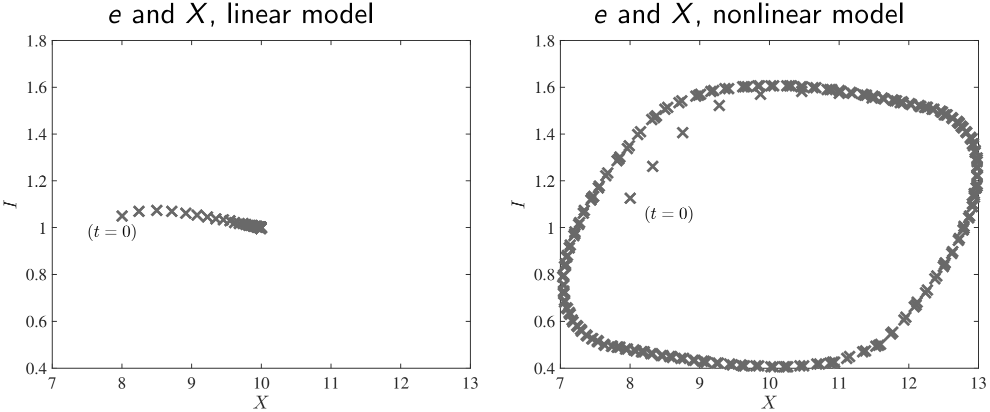


FIGURE 30 – Deterministic simulation

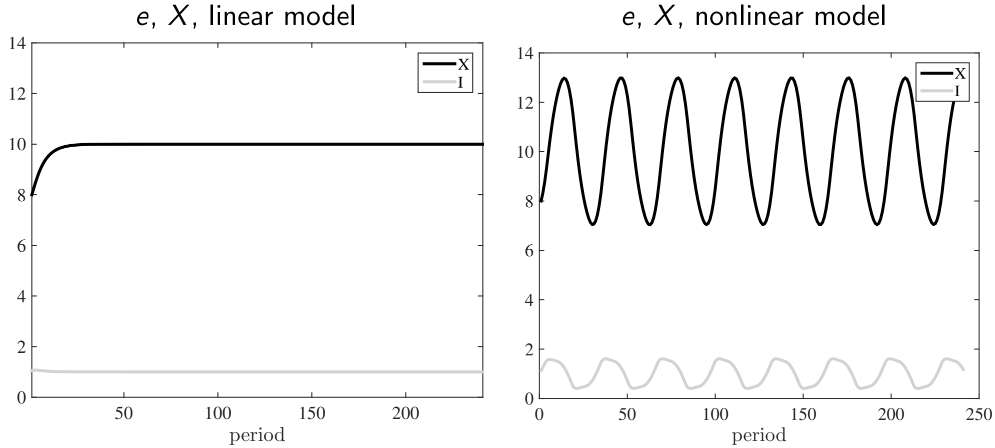


FIGURE 31 – One stochastic simulation

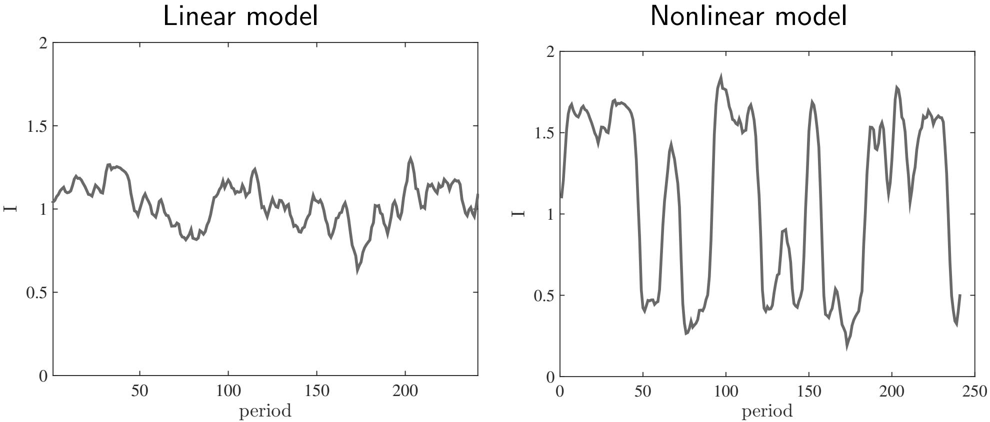


FIGURE 32 – $e_t = \sin(\omega t)$

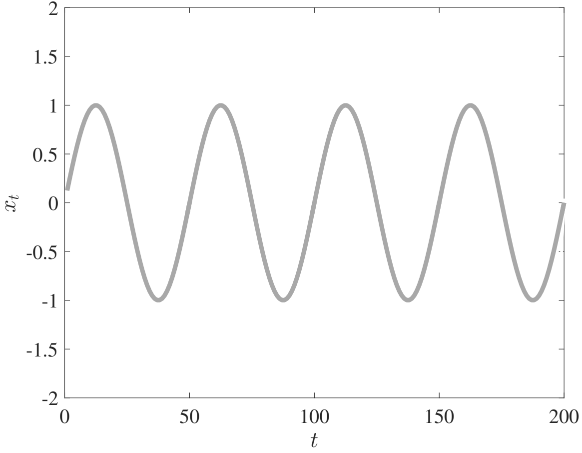


FIGURE 33 – What the results are not : $e_t = \sin(\omega t) + u_t$

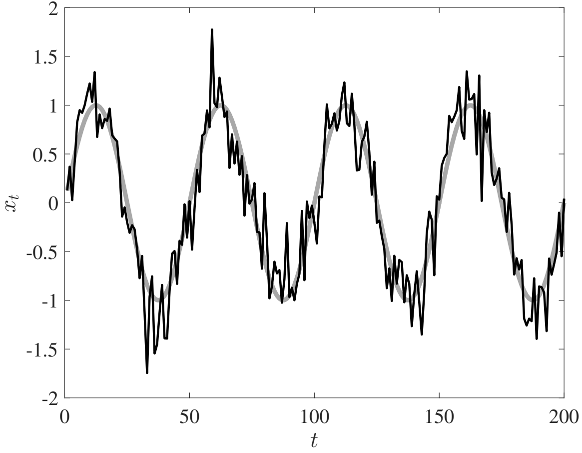


FIGURE 34 – What the results are :

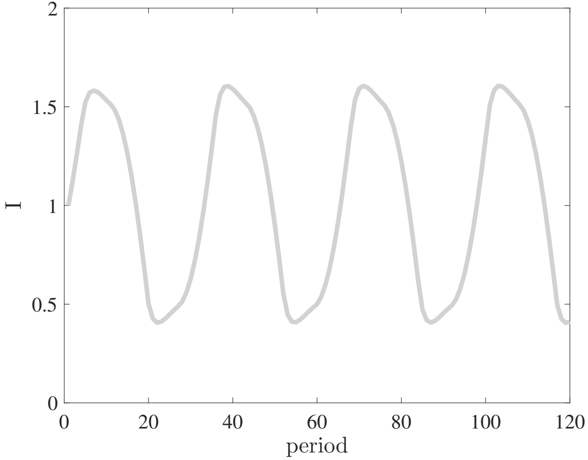
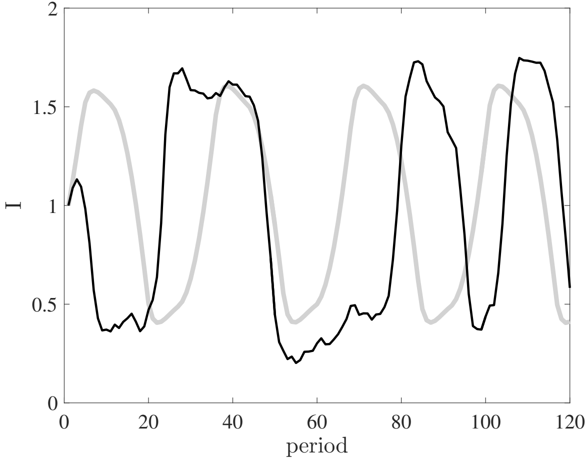


FIGURE 35 – What the results are :



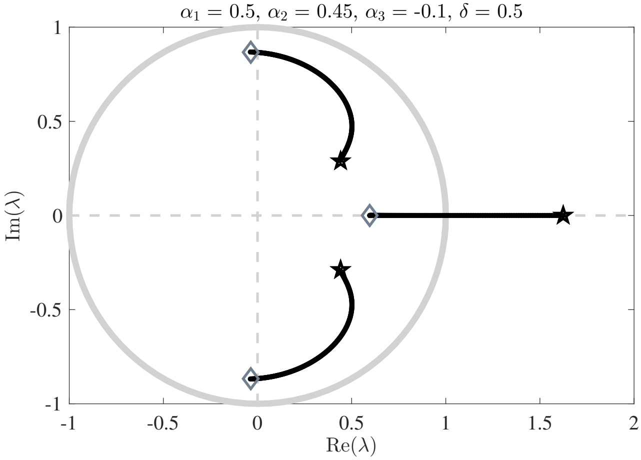
- Cycles were not a consequence of equilibrium selection with rational expectations (not like sunspots)
- But robust to rational expectations.
- $$e_{it} = \alpha_0 - \alpha_1 X_{it} + \alpha_2 e_{it-1} + \alpha_3 E_t[e_{it+1}] + F\left(\frac{\sum e_{jt}}{N}\right)$$
- with accumulation remaining the same

$$X_{it} = (1-\delta)X_{it} + e_{it}$$

- Restrict attention to situations where this system is saddle path stable absent of complementarities.
- The local dynamics is described by the 3 eigenvalues of the linearized system

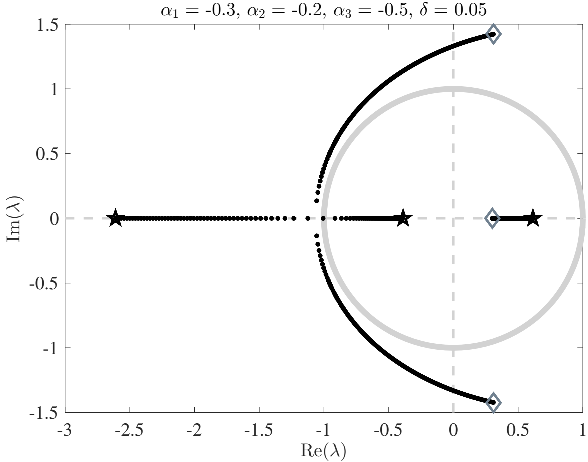
- Initial situation has two stable roots and one unstable
- Three types of bifurcations are possible :
  1. The unstable root enters the unit circle : local indeterminacy arises ("Benhabib-Farmer ")
  2. One stable root leaves the unit circle : instability arises with a flip or fold type bifurcation
  3. Two stable roots leave the unit circle simultaneous because they are complex : this is a Hopf bifurcation

FIGURE 36 – Eigenvalues of the Reduced Form Model



- Initial situation has two stable roots and one unstable
- Three types of bifurcations are possible :
  1. The unstable root enters the unit circle : local indeterminacy arises ("Benhabib-Farmer ")
  2. One stable root leaves the unit circle : instability arises with a flip or fold type bifurcation
  3. Two stable roots leave the unit circle simultaneous because they are complex : this is a Hopf bifurcation

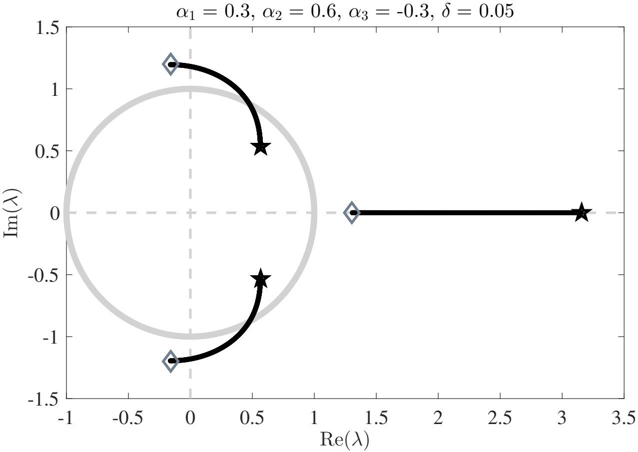
FIGURE 37 – Eigenvalues of the Reduced Form Model



- Initial situation has two stable roots and one unstable
- Three types of bifurcations are possible :
  1. The unstable root enters the unit circle : local indeterminacy arises ("Benhabib-Farmer ")
  2. One stable root leaves the unit circle : instability arises with a flip or fold type bifurcation
  3. Two stable roots leave the unit circle simultaneous because they are complex : this is a Hopf bifurcation

Set of potential bifurcation with Forward looking elements

Figure 38 – Eigenvalues of the Reduced Form Model



Proposition 10

If unique steady state, then no indeterminacy nor Fold bifurcations (always one positive and greater than one eigenvalue)

Proposition 11

If $\alpha_1 > 0$ and α_2 (sluggishness) sufficiently large, then the two other eigenvalues will become complex and

- *either will stay inside the unit disk (non locally explosive cycles)*
- *or will exit the unit disk (* HOPF *bifurcation and limit cycles)*

FIGURE 39 – Increasing complementarities ρ : local stability

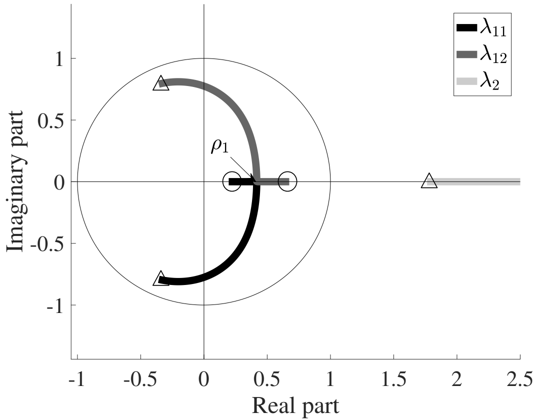


FIGURE 40 – Increasing complementarities ρ : local instability and limit cycle

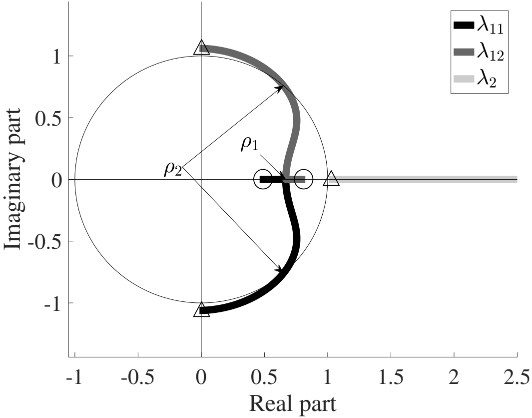


FIGURE 41 – A Saddle Limit Cycle

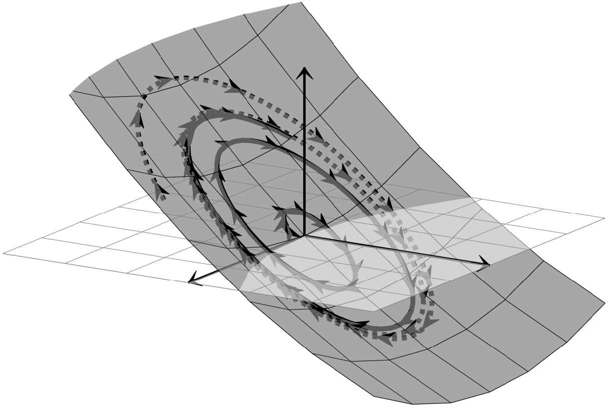


FIGURE 42 – Explosive path

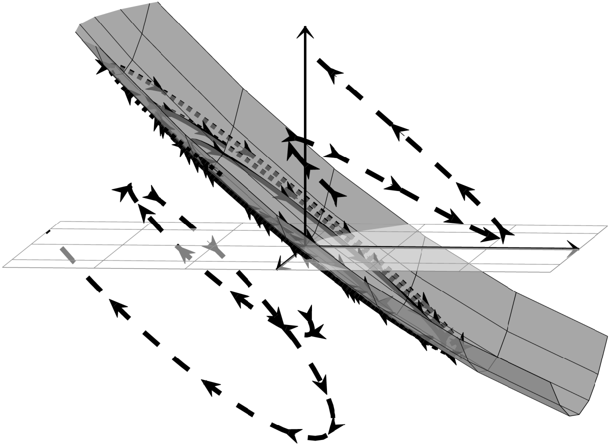
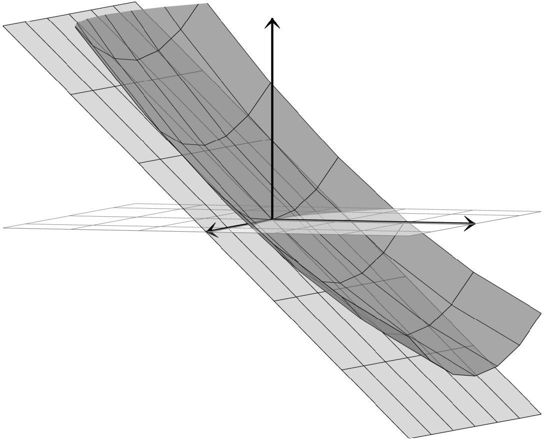


FIGURE 43 – Linear and non linear stable manifolds



Roadmap

I. Cyclicality
II. Instability
III. Generating Cycles through Dynamic Models with Interactions
IV. A Fully Specified Model

- Stylized NK model which is extended to allow for the forces highlighted in our general structure.
- We add accumulation of durable-housing goods and habit persistence : *accumulation and sluggishness*
- Financial frictions imply a counter-cyclical risk premium : *complementarities*
- Estimate parameters based on spectrum observations and higher moments. (use perturbation method and indirect inference)

1. Household buy consumption services to maximise utility taking prices as given
2. Firms supply consumption services to the market where the services can come from existing durable goods or new production.
3. These firms have sticky prices.
4. Central Bank set policy rate according to a type of Taylor rule
5. Interest rate faced by households is the policy rate plus a risk premium, where the risk premium varies with the cycle.
   - unemployed workers may default
   - To break even, banks charge a risk premium
   - More aggregate consumption $\rightsquigarrow$ more employment $\rightsquigarrow$ less default $\rightsquigarrow$ lower risk premium $\rightsquigarrow$ more individual consumption
   - This creates complementarity : if the rest of the economy consumes more, the risk premium falls $\rightsquigarrow$ I tend to consume more.

- Solution is

$$\ell_t = \mu_t + \widehat{\alpha}_1 X_t + \widehat{\alpha}_2 \ell_{t-1} + \widehat{\alpha}_3 \mathbb{E}_t \left[\ell_{t+1}\right] + \widehat{F}(\ell_t)$$

- together with accumulation

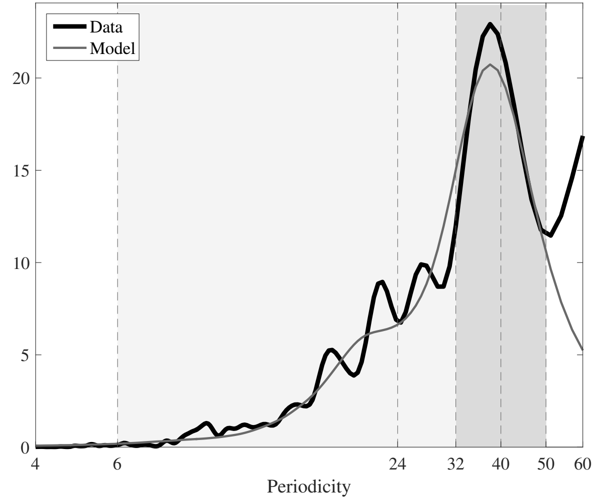
$$X_{t+1} = (1 - \delta) X_t + \psi \ell_t$$

- Shock
  - × AR(1) discount factor shock (μ_t)
- Observables
  - × ℓ_t is (log) employment (and also output gap),
  - × Risk Premium : Fed Funds Rate - BAA Bonds spread.

- Estimate parameters of model by Indirect Inference
- Targets
  - spectrum of hours worked on the frequencies 2-50
  - spectrum of interest rate spread on the frequencies 2-50
  - Set of other higher moments (correlation, kurtosis and skewness of hours and spread)

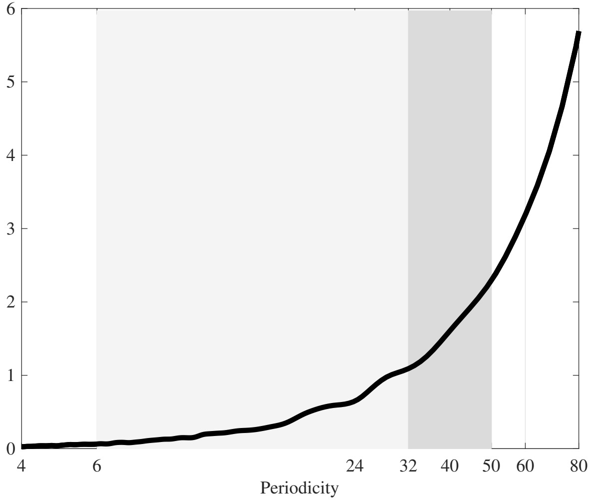
IV. A Fully Specified Model

Spectrum fit for Hours



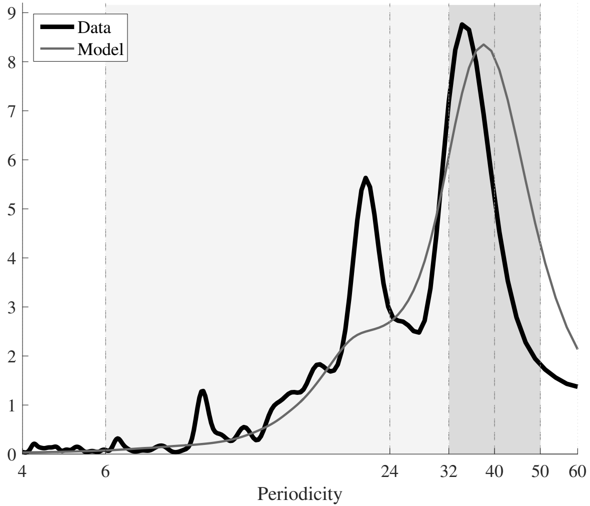
IV. A Fully Specified Model

Hours Spectrum in Smets & Wouters' Model



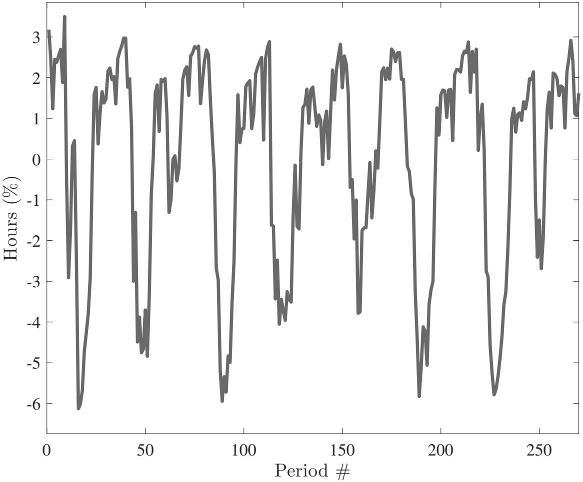
IV. A Fully Specified Model

Spectrum fit for Spread



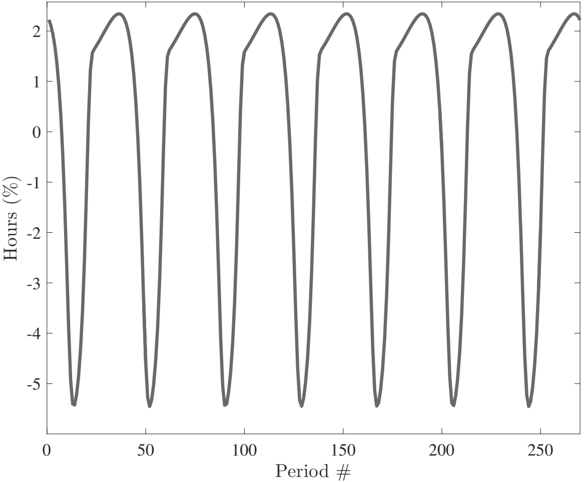
IV. A Fully Specified Model

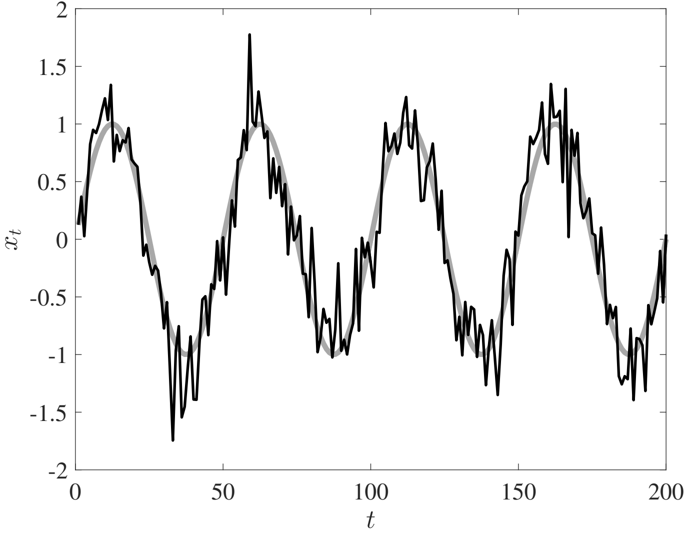
Sample Draw for Hours

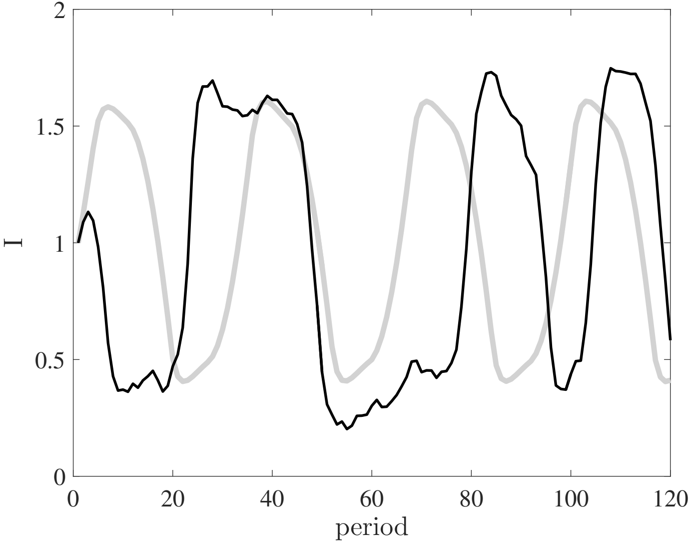


IV. A Fully Specified Model

Sample Draw for Hours, no shocks

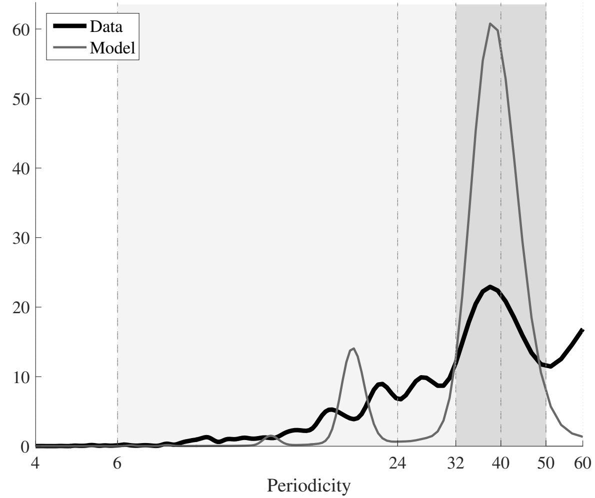



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
x_t
0
50
100
150
200
t


2
1.5
1
0.5
0
I
0
20
40
60
80
100
120
period

IV. A Fully Specified Model

Spectrum for Hours, no shocks



IV. A Fully Specified Model

Shocks : $\mu_t = \rho\mu_{t-1} + \varepsilon_t$

TABLE 2 – Estimated Parameter Values

...	...
...	...
ρ	-0.0000
...	...
...	...

- Shocks are important in our framework for explaining the data
- But they are *i.i.d.*
- Hence, almost all dynamics are internal.

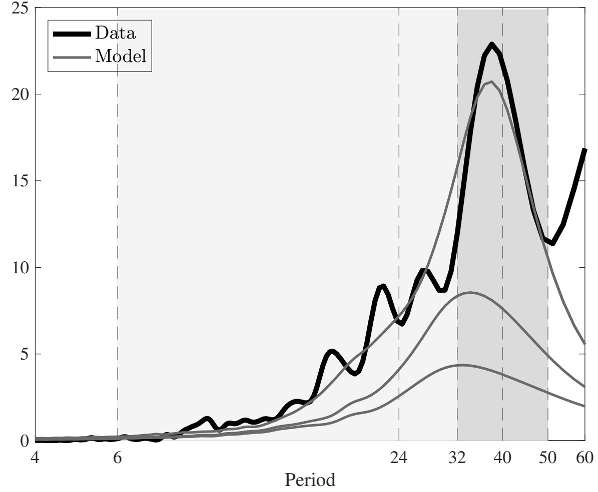
- Policy rule :

$$i_t = \rho^N + E_t \pi_{t+1} + \phi_\ell E_t \ell_{t+1}$$

- Let's increase ϕ_ℓ
- "Cyclical" policy has strong effect on the "structural" forces that shape the cycle.

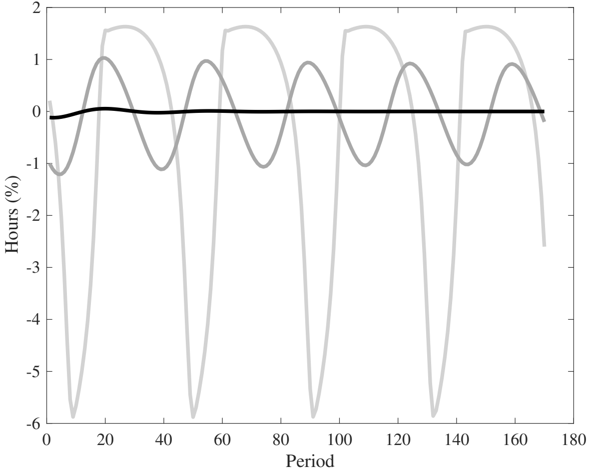
IV. A Fully Specified Model

Policy experiment - Hours Spectrum, Increasing ϕ_ℓ



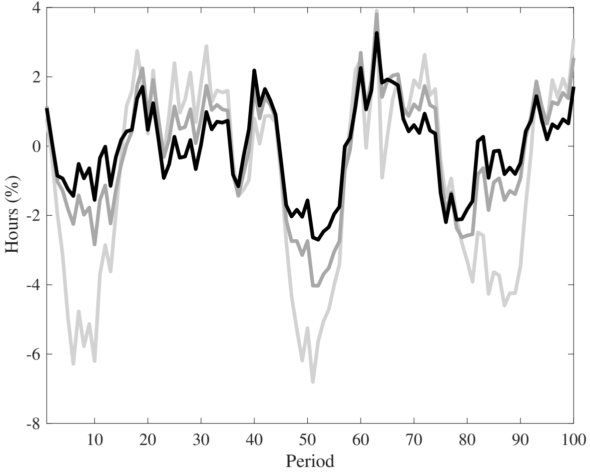
IV. A Fully Specified Model

Policy experiment -Hours Deterministic Simulation, Increasing ϕ_ℓ



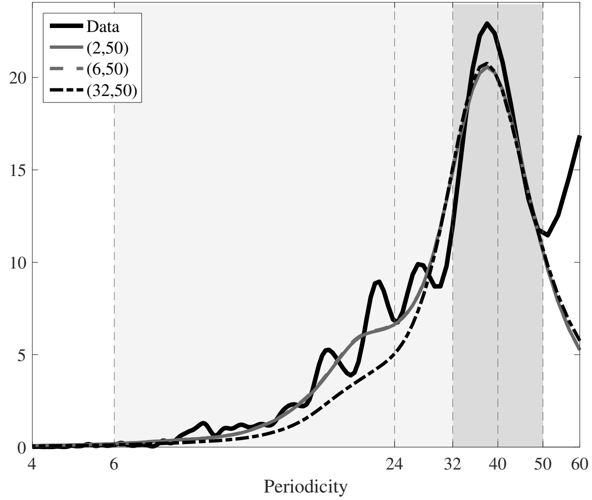
IV. A Fully Specified Model

Policy experiment - Hours, One Stochastic Simulation, Increasing ϕ_ℓ



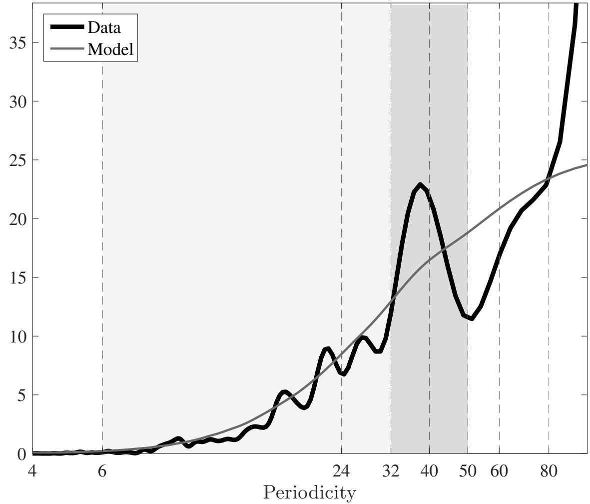
IV. A Fully Specified Model

Estimating to Match Spectral Density over (x,50) (Hours)



IV. A Fully Specified Model

Estimating to Match Spectral Density over (2,100) (Hours)



Hours Impulse Response at Cycle Peak